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On the Transition Dynamics in Endogenous Recombinant Growth Models*

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Abstract

This paper constitutes a first attempt at studying the transition dynamics of the Tsur and Zemel (2007) continuous time endogenous growth framework in which knowledge evolves according to the Weitzman (1998) recombinant process. For a specific choice of the probability function characterizing the Weitzman recombinant process, we find a suitable transformation for the state and control variables in the dynamical system diverging to asymptotic constant growth, so that an equivalent 'detrended' system converging to a steady state in the long run can be tackled. Since the dynamical system obtained so far turns out to be analytically intractable, we rely on numerical simulation in order to fully describe the transition dynamics for a set of values of the parameters.

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1 Introduction

Tsur and Zemel (2007) developed an endogenous growth model in which balanced long-run growth is obtained by assuming that the stock of knowledge evolves according to Weitzman's (1998) recombinant expansion process and is used, together with physical capital, as input factor by competitive firms in order to produce a unique physical good. At each instant new knowledge is produced by an independent R&D sector directly controlled by a 'regulator' who aims at maximizing the discounted utility of a representative consumer over an infinite horizon. The optimal resources required for new knowledge production are obtained by the regulator in the form of a tax levied on the consumers. The economy, thus, envisages two sectors, a competitive one devoted to the production of the unique physical good, and a regulated R&D sector in which the public good 'knowledge' is being directly financed by the regulator and produced according to Weitzman's production function.

In such framework Tsur and Zemel provide conditions under which the economy performs sustained constant balanced growth in the long run; moreover, when balanced growth occurs, they also characterize the asymptotic optimal tax rate and the common growth rate of all variables. Hence,

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by endogenizing the optimal choice for investing in knowledge production, their result generalizes Weitzman (1998) endogenous growth model in which the investment in knowledge production was assumed to be constant and exogenously determined.

In this paper we further extend the Tsur and Zemel results by studying more accurately the transition dynamics along a characteristic turnpike curve in the knowledge-capital state space already discussed in Tsur and Zemel (2007). For a specific parametrization of the model and when the conditions allowing sustained long-run growth are met, we are able to (numerically) compute the optimal policy – in terms of optimal consumption – and thus the optimal time-path trajectories of the stock of knowledge, capital, output and consumption – as well as their transition growth rates – while the economy is being headed along the turnpike curve toward its long-run constant balanced growth behavior.

Our method is based on the standard technique of transforming the state and control variables of the Hamiltonian describing the optimal dynamics of (a slightly generalized version of) the Tsur and Zemel model – all diverging in the long-run – into 'detrended' state-like and control-like variables, both converging to a saddle-path stable steady state in the appropriate space as time elapses. To study such detrended system we apply the time-elimination method introduced by Mulligan and Sala-i-Martin (1991) (see also Mulligan and Sala-i-Martin, 1993, and Barro and Sala-i-Martin, 2004, pp. 593-596) so that the optimal detrended consumption policy can be calculated by means of numerical methods for ODEs; then, substituting such policy in the ODE of the state-like variable and solving it – again numerically – with respect to time, the optimal time-path trajectories of both state-like and control-like variables are obtained. Eventually, these trajectories are reconverted into time-path trajectories for the original model, thus allowing for a detailed analysis of the transition dynamics of all relevant variables.

Two main technical difficulties had to be overcome: 1) finding a proper probability function for the Weitzman's recombinant process suitable for the change of variables in the construction of the detrended system of ODEs, and 2) the exploitation of a singular point – other than the saddle-path steady state – along the turnpike curve, which can be used as initial condition for calculating specific solutions for the ODE describing the policy. Due to the high instability of the system of ODEs characterizing the detrended variables, we have been able to fully solve the model only for a set of values of the parameters; more precisely, our approach works satisfactory only on a manifold of dimension one in the parameters' space (see Remark 1 at the end of Section 6).

In Section 2 the original contribution by Weitzman (1998) on the production of new knowledge by combining existing ideas – and its adaptation to a continuous time setting – is briefly recalled. Section 3 introduces an endogenous recombinant growth model based on the framework provided by Tsur and Zemel (2007) and recalls the main asymptotic results known for this model, while Section 4 better specifies the dynamics along a transitional turnpike. The central contribution of this paper is contained in Section 5, where, under a suitable choice for the functions of the model – in particular, for the Weitzman probability of success in matching pairs of ideas – we are able to transform the original diverging dynamics into an equivalent system of two ODEs in two 'detrended' variables converging asymptotically to a steady state in the appropriate space. This allows for numeric computation of the optimal policy of both the detrended system and the original diverging dynamics, which is implemented in Section 6 for a specific set of parameters' values. Finally, after using the optimal policy obtained so far to numerically trace out the optimal time-path trajectories, Section 7 is dedicated to a qualitative discussion of the transition dynamics thus obtained, while Section 8 reports some concluding remarks and topics for future research.

2 Recombinant growth

2.1 Producing ideas by means of ideas

Weitzman (1998) stylizes the production of knowledge through a function that uses previous knowledge inputs and exhibits 'strongly' increasing returns. Weitzman's device postulates that originally unprocessed ideas, *seed* in his terminology, are blended with all other ideas available in order to generate new *hybrid* seed ideas; a costly selection process permits in turn to extract from those a subset of *fertile* seed ideas that are again recombined with all the existent fertile ideas to produce yet new hybrids, and so on. Therefore the process occurs indefinitely, generating knowledge growth.

The hybridization is based on matching m ideas together and then checking whether such matching is able to produce a new fertile (*i.e.*, successful) idea. If $A(t)$ is the stock of knowledge available at time t (measured as the total number of fertile ideas), let $C_m[A(t)]$ denote the number of different combinations of m elements (hybrids) of $A(t)$; *i.e.*:

$$C_m[A(t)] = \binom{A(t)}{m} = \frac{A(t)!}{m!\,[A(t)-m]!}.$$

If $m=2$, $C_2(A) = A(A-1)/2$, while, if $m=3$, $C_3(A) = A(A-1)(A-2)/[6(A-3)]$, and so on. Therefore, at time t the number of hybrid seed ideas is given by

$$H(t) = C_m[A(t)] - C_m[A(t-1)]. \tag{1}$$

By assuming a probability π of obtaining a successful idea from each hybridization (matching), the number of new successful idea generated by $H(t)$ seed ideas at any given time t is given by [see eqn. (2) on p. 337 in Weitzman, 1998]:

$$\Delta A(t) = A(t+1) - A(t) = \pi H(t) = \pi\left\{C_m[A(t)] - C_m[A(t-1)]\right\}, \tag{2}$$

which, in a discrete time framework, defines a *recombinant expansion process* of second order. It represents the potential knowledge production path.

According to (2), the stock of knowledge A has the potential of growing faster than exponentially, that is, at an increasing rate of growth (Lemma on p. 338 in Weitzman, 1998). However, since the hybridization process of seed ideas, as previously asserted, necessarily consumes an amount of physical resources, potentially explosive growth is precluded by physical constraints; precisely, scarcity of resources. As a matter of fact, Weitzman (1998) shows that knowledge actually grows at some bounded positive rate, thus reconciling his theory with standard endogenous growth models, suggesting that the growth rate of GNP in real economies should be bounded as well (see, *e.g.*, Romer, 1996, Aghion and Howitt, 1999, or Barro and Sala-i-Martin, 2004). Accordingly, the knowledge generation mechanism envisaged by Weitzman uses two inputs: hybrid seed ideas H, in the fashion already discussed, and physical resources J. The latter, although not entering directly the recombinant process, affects the probability π of producing successful ideas – *i.e.*, transforming hybrid seeds in fertile seeds – so that π turns out to be increasing in J for each given H. However, a fixed amount of resources J becomes less productive if hybrid seed ideas H increase. To summarize, the success probability π results to be increasing in the ratio J/H.

All these considerations lead to the following *production function for new knowledge* ΔA which uses the two variables H and J as input factors:

$$\Delta A = W(J,H) = H\pi\left(\frac{J}{H}\right), \tag{3}$$

which corresponds to (28) on p. 346 in Weitzman (1998). Note that $W(\cdot,\cdot)$ in (3) is homogeneous of degree 1 in the variables J and H. In the sequel we shall assume the following.

A. 1 *The function* $\pi : \mathbb{R}_+ \rightarrow [0,1]$ *is independent of time and is such that* $\pi' > 0$, $\pi'' < 0$, $\pi(0) = 0$ *and* $\pi(\infty) \leq 1$*; moreover, it will be assumed that*[1] $\lim_{x\rightarrow 0^+} \pi'(x) < +\infty$.

Provided that the resources J employed in the production of new knowledge are a constant fraction of the total output y produced by the economy, $J = sy$, where s is exogenously determined, Weitzman (1998) establishes that in the long run the asymptotic growth rate is a positive constant which depends on the saving rate s.

2.2 The continuous time setting

In a recent work, Tsur and Zemel (2007), made an important refinement of Weitzman's analysis by endogenizing the (optimal) determination of the resources J employed in the production of new knowledge.[2] Their model features a 'regulator', a sort of Leviathan, who has the task of choosing the optimal amount J to be employed into the production of new knowledge – which, in turn, is being assigned to all firms producing the amount y of a unique (physical) output – in order to maximize the discounted utility of a representative consumer over an infinite horizon. Output producing firms operate in a competitive environment, while the regulator has the power to levy the exact amount J as a tax on the representative consumer, through which, given all the H hybrid seed ideas freely available, new useful knowledge is being directly generated according to (3), $\Delta A = H\pi(J/H)$, and is immediately and freely passed to the output producing firms.

The difficulty in dealing with the second-order dynamic (2) in the constraint of the maximization problem is overcome by switching from the Weitzman's discrete time formulation into a continuous time model. This allows the authors to rewrite (1) as follows:

$$H(t) = C'_m [A(t)] \dot{A}(t), \tag{4}$$

where $\dot{A}(t)$ denotes the derivative of the stock of knowledge at instant t, $A(t)$, with respect to time t, and corresponds to $\Delta A(t)$ in the discrete time framework. By replacing $\Delta A(t)$ with $\dot{A}(t)$ in (3) we obtain the analogous of Weitzman's new knowledge production function, (3), in continuous time:

$$\dot{A}(t) = H(t)\,\pi\left[\frac{J(t)}{H(t)}\right], \tag{5}$$

where the probability of generating a new fertile idea π still satisfies A.1.

By combining (4) and (5) the following law of motion for the stock of knowledge $A(t)$ is obtained:

$$\dot{A}(t) = \frac{J(t)}{\varphi[A(t)]}, \tag{6}$$

where

$$\varphi(A) = C'_m(A)\,\pi^{-1}\left[\frac{1}{C'_m(A)}\right] \tag{7}$$

is the *expected unit cost of knowledge production*. Note that $\varphi(\cdot)$ is decreasing and, as knowledge keeps spreading, it converges to

$$\lim_{A\rightarrow\infty} \varphi(A) = \frac{1}{\pi'(0)} > 0, \tag{8}$$

where $1/\pi'(0)$ is strictly positive by Assumption 1.

[1] For simplicity, in the sequel $\lim_{x\rightarrow 0^+} \pi'(x)$ will be denoted by $\pi'(0)$.

[2] Here our analysis slightly departs from the original model by Tsur and zemel by allowing J to be any amount of physical capital available in the economy, while the authors constrain such resources to be only a fraction $0 \leq s \leq 1$ of the total output y. In other words, in our economy the regulator has the power to extract resources also from existing physical capital, in addition to the whole total output y.

3 Endogenous recombinant growth

With no loss of generality, in the sequel we shall assume that labour is constant through time and normalized to one:[3] $L \equiv 1$. The output producing firms use a neoclassical production function,

$$y(t) = F[k(t), A(t)], \tag{9}$$

depending on aggregate capital, k, and knowledge-augmented labour [$A(t)L$, with $L = 1$].

A. 2 $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *exhibits constant returns to scale and is such that* $F_k > 0$, $F_A > 0$, $F_{kk} < 0$, $F_{AA} < 0$, $F_{kA} > 0$, *and satisfies the Inada condition* $\lim_{k \to 0^+} F(k, A) = +\infty$ *for all* $A > 0$.

Each firm i maximizes instantaneous profit by renting capital k_i and hiring labour $L_i \leq 1$ from the households, while taking as given the capital rental rate r, the labour wage w and the stock of knowledge A. Under the assumption that all firms use the same technology and operate in a competitive market, and that all households are the same, the subscript i can be dropped and (9) can be rewritten as $y = Af(k/A)$, where

$$f(x) = F(x, 1). \tag{10}$$

Since firms act competitively, in equilibrium their profit is zero, that is, households earn $y = Af(k/A) = rk + w$; moreover, the amount of capital demanded, k, satisfies

$$f'(k/A) = r. \tag{11}$$

Given that a fraction $J(t)$ of the whole endowment of the economy, $k(t) + y(t)$, is being employed to finance R&D firms, and a fraction $c(t)$ is being consumed, capital evolves through time according to

$$\dot{k}(t) = y(t) - J(t) - c(t), \tag{12}$$

where $c(t)$ denotes instantaneous per capita consumption and, for simplicity, it is assumed that capital does not depreciate. Since the upper bound[4] for $J(t)$ and $c(t)$ is jointly given by $J(t) + c(t) \leq k(t) + y(t)$, $\dot{k}(t)$ in (12) may be negative.

Assuming that all households enjoy an instantaneous utility $u[c(t)]$, with $u : \mathbb{R}_+ \to \mathbb{R}_+$ increasing and strictly concave, the 'regulator' solves

$$\max_{\{c(t), J(t)\}} \int_0^\infty u[c(t)] e^{-\rho t} dt \tag{13}$$

$$\text{subject to} \begin{cases} \dot{A}(t) = J(t) / \varphi[A(t)] \\ \dot{k}(t) = F[k(t), A(t)] - J(t) - c(t) \\ J(t) + c(t) \leq k(t) + F[k(t), A(t)] \\ k(t) \geq 0, J(t) \geq 0, c(t) \geq 0 \\ k(0) = k_0 > 0, A(0) = A_0 > 0, \end{cases}$$

where utility is discounted at a constant rate $\rho > 0$. (13) may be interpreted as a maximum welfare problem, where k and A are the state variables and c and J are the controls; the regulator chooses the

[3]Tsur and Zemel (2007) assume that the amount of labour available in the economy is L, constant through time even if not necessarlily equals to one. As stationarity with respect to time of L is the strong assumption here, normalizing labour to $L \equiv 1$ has the advantage of simplifying notation at no cost.

[4]See note 2.

optimal consumption, $\{c(t)\}$, and the optimal investment in R&D, $\{J(t)\}$, policies by taking into account the evolution of knowledge according to (6).

Suppressing the time argument, the current-value Hamiltonian associated to (13) is

$$H(A,k,J,c,\vartheta_1,\vartheta_2) = u(c) + \vartheta_1 [F(k,A) - J - c] + \vartheta_2 \frac{J}{\varphi(A)}, \tag{14}$$

where A and k are the state variables, c and J are the controls, ϑ_1 and ϑ_2 are the costate variables associated with k and A respectively. Necessary conditions are the following:

$$u'(c) = \vartheta_1 \tag{15}$$

$$J = \begin{cases} 0 & \text{if } \vartheta_2/\varphi(A) < \vartheta_1 \\ \tilde{J} & \text{if } \vartheta_2/\varphi(A) = \vartheta_1 \\ k + F(k,A) - c & \text{if } \vartheta_2/\varphi(A) > \vartheta_1 \end{cases} \tag{16}$$

$$\dot{\vartheta}_1 = \rho\vartheta_1 - \vartheta_1 F_k(k,A) \tag{17}$$

$$\dot{\vartheta}_2 = \rho\vartheta_2 - \vartheta_1 F_A(k,A) + \vartheta_2 \frac{J\varphi'(A)}{[\varphi(A)]^2} \tag{18}$$

$$\lim_{t\to\infty} H(t)\, e^{-\rho t} = 0, \tag{19}$$

where $\tilde{J}$ in (16) will be defined later in (22). Clearly, the case $J = k + F(k,A) - c$ when $\vartheta_2/\varphi(A) > \vartheta_1$ in (16) can be immediately ruled out by the Inada condition of Assumption A.2; therefore, $\vartheta_2/\vartheta_1 \leq \varphi(A)$, must hold.

By differentiating $\vartheta_1 = \vartheta_2/\varphi(A)$ in (16) with respect to time and coupling it with (17) and (18), the following condition is met:

$$F_k(k,A) - \frac{F_A(k,A)}{\varphi(A)} = 0, \tag{20}$$

defining the locus on the state space (A,k) on which the marginal product of capital equals that of knowledge per unit cost. Equation (20) can be rewritten as $z(k/A) = \varphi(A)$ where $z(x) = f(x)/f'(x) - x$, with f defined in (10), is an increasing function of x; thus, the curve defined by (20) can be expressed as a function of the only variable A:

$$\tilde{k}(A) = z^{-1}[\varphi(A)]\, A, \tag{21}$$

where z^{-1} is the inverse of $z(x)$.

Differentiating $\tilde{k}(A)$ with respect to time, substituting into (12) and using (6) yields

$$\tilde{J}(t) = [y(t) - c(t)] \frac{\varphi[A(t)]}{\tilde{k}'[A(t)] + \varphi[A(t)]}, \tag{22}$$

where $y(t) = F[k(t), A(t)]$. Condition (22) establishes a relationship between the optimal investment in R&D, $\tilde{J}(t)$, as a function of the other control variable, the optimal consumption $c(t)$, when the economy is constrained to grow along the curve $\tilde{k}(A)$ defined in (21); that is, in view of (16), when $\vartheta_2(t)/\varphi[A(t)] = \vartheta_1(t)$ holds.

It will be useful to consider the limiting shape of (21), which, for larger A, tends to become linear. For this purpose, define its asymptote:

$$\tilde{k}_\infty(A) = \tilde{\eta}A + q, \tag{23}$$

where, using (8), $\tilde{\eta} = z^{-1}[1/\pi'(0)]$ and q is a non-negative constant. Note that $\tilde{k}(A)$ lies above $\tilde{k}_\infty(A)$ for all $A < \infty$, and approaches $\tilde{k}_\infty(A)$ as A increases. Whether the intercept q is zero or strictly positive depends on the number of ideas m being matched at each instant t in Weitzman's recombinant process (4).

Proposition 1 *The intercept q in (23) is zero whenever $m > 2$, while $q > 0$ for $m = 2$.*

Proof. Since $\tilde{k}_\infty(A) = \tilde{\eta}A + q$ is the asymptote of $\tilde{k}(A)$,

$$q = \lim_{A\to+\infty}\left[\tilde{k}(A) - \tilde{\eta}A\right] = \lim_{A\to+\infty}\left\{z^{-1}[\varphi(A)] - z^{-1}[1/\pi'(0)]\right\}A. \tag{24}$$

As $\varphi(A)$ is decreasing and, under Assumption A.1, bounded away from zero [specifically $0 < 1/\pi'(0) \le \varphi(A) \le \varphi(A_0)$], by Assumption A.2 the term $z^{-1}[\varphi(A)] - z^{-1}[1/\pi'(0)]$ in (24) is $o[\varphi(A)]$. Thus, since, by (7), $O[\varphi(A)] = O[C'_m(A)] = O(A^{m-1})$ [*i.e.*, $C'_m(A) \sim A^{m-1}$ for large A], for $m > 2$ the limit in (24) is zero, while, for $m = 2$, such limit must be nonzero; as $z^{-1}[\varphi(A)] - z^{-1}[1/\pi'(0)] > 0$ for all $A < +\infty$, $q > 0$ must hold whenever $m = 2$. ■

Another locus in the state space will be used in the analysis: the curve on which the marginal product of capital equals the individual discount rate, $f'(k/A) = \rho$, which, by (11), implies $r = \rho$. As $f'(k/A)$ is decreasing, also such curve can be expressed as a function of real variable:

$$\hat{k}(A) = \hat{\eta}A, \tag{25}$$

with $\hat{\eta} = (f')^{-1}(\rho)$; that is, $\hat{k}(A)$ is the linear function with slope $\hat{\eta} > 0$.

The curves $\tilde{k}(A)$, $\tilde{k}_\infty(A)$ and $\hat{k}(A)$ defined in (21), (23) and (25) respectively, will be labelled *turnpike*, *asymptotic turnpike* and *stagnation line* respectively. The optimal investment in R&D, $\tilde{J}(t)$, when the economy grows along the turnpike $\tilde{k}(A)$ defined by (22) will be referred as the *singular policy*.

In order to simplify our analysis, throughout the whole paper we shall assume the following.

A. 3 *The instantaneous utility of the representative consumer is of the CIES type:*

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma},$$

with the reciprocal of the intertemporal elasticity of substitution satisfying $\sigma \ge 1$.

The next proposition summarizes the main results in Tsur and Zemel (2007).

Proposition 2 (Tsur and Zemel)

i) *A necessary condition for the economy to sustain long-run growth is*

$$\hat{\eta} > \tilde{\eta}; \tag{26}$$

conversely, if $\hat{\eta} \le \tilde{\eta}$ the economy eventually reaches a steady (stagnation) point on the line (25) corresponding to zero growth.

ii) *Under condition (26), for any given initial knowledge stock A_0 there is a corresponding threshold capital stock $k^{sk}(A_0) \ge 0$ such that whenever $k_0 \ge k^{sk}(A_0)$ the economy – possibly after an initial transition outside the turnpike $\tilde{k}(A)$ – first reaches the turnpike $\tilde{k}(A)$ defined in (21) in a finite time, and then continues to grow along it as time elapses until the asymptotic turnpike $\tilde{k}_\infty(A)$ defined in (23) is reached in the long-run. Along $\tilde{k}_\infty(A)$ the economy follows a balanced growth path characterized by a common constant growth rate of output, knowledge, capital and consumption defined by:*

$$\gamma = \frac{r_\infty - \rho}{\sigma} > 0, \tag{27}$$

where $r_\infty = \lim_{A\to\infty} f'\left[\hat{k}(A)/A\right] = \lim_{A\to\infty} f'\left[\hat{k}_\infty(A)/A\right] = f'(\tilde{\eta})$ *defines the long-run capital rental rate.*[5] *Moreover,* $\tilde{J}(t) < y(t) = F[k(t), A(t)]$ *for large* t*, and the income shares devoted to investments in knowledge and capital are constant and given respectively by*

$$s_\infty = \frac{\gamma}{r_\infty}\left(\frac{1}{1+\tilde{\eta}\pi'(0)}\right) \qquad \text{and} \qquad s_\infty^k = \frac{\gamma}{r_\infty}\left(\frac{\tilde{\eta}\pi'(0)}{1+\tilde{\eta}\pi'(0)}\right). \tag{28}$$

If $k_0 < k^{sk}(A_0)$ *the economy eventually stagnates.*

Proposition 2, whose proof can be found in the Appendix in Tsur and Zemel (2007), establishes that, if (26) holds and initial capital stock k_0 is sufficiently high (with respect to initial knowledge stock A_0), the economy is able to grow along a turnpike path which, in the long run, converges to a balanced growth path in which both knowledge and physical capital grow at the same positive constant rate and the saving rate is positive and constant as well, thus confirming the original Weitzman result in a more general setting.

As we ruled out the case $\vartheta_2/\varphi(A) > \vartheta_1$ in (16), two (optimal) regimes are possible:

1. zero R&D, corresponding to $J \equiv 0$, which, if maintained forever, eventually leads the economy to some steady state (stagnation point) on the line $\hat{k}(A)$ defined in (25), and

2. an optimal path along the turnpike $\tilde{k}(A)$ defined in (21) – maybe started after a finite period of transition outside the turnpike itself – corresponding to the singular policy $\tilde{J}$ satisfying (22), which envisages growth for all variables as time elapses and, if maintained forever, eventually lead to a balanced growth path along the asymptotic turnpike $\tilde{k}_\infty(A)$ defined in (23).

Since, under conditions (26) and $k_0 \geq k^{sk}(A_0)$, it can be shown that the turnpike $\tilde{k}(A)$ is 'trapping'– *i.e.*, the economy keeps growing along the turnpike whenever it reaches it by selecting the optimal policy $\tilde{J}$ as in (22) thereafter – there are two types of transition dynamics: the first driving the system toward the turnpike starting from some initial condition outside it, and the second characterizing the optimal path along the turnpike $\tilde{k}(A)$ after the economy entered it. In the sequel we shall focus on the latter: specifically, we shall assume that (26) holds, that is, $\hat{\eta} > \tilde{\eta}$, which implies that the line containing all potential steady states for the economy, $\hat{k}(A)$, must lie strictly above[6] the turnpike $\tilde{k}(A)$ on the state space (A, k) for A sufficiently large; moreover, we shall restrict our attention to initial conditions A_0 and k_0 such that $k_0 = \tilde{k}(A_0)$. Note that in this scenario the *Skiba condition* $k_0 \geq k^{sk}(A_0)$ is certainly satisfied, as the turnpike $\tilde{k}(A)$ is trapping.

4 Dynamics along the turnpike

We now adapt the optimal conditions (15) - (19) to the system's behavior along the turnpike $\tilde{k}(A)$. All variables on the turnpike will be labelled with a '$\sim$' symbol.

Suppressing the time argument and using (22), (6) becomes

$$\dot{A} = \frac{\tilde{y}(A) - \tilde{c}}{\tilde{k}'(A) + \varphi(A)}, \tag{29}$$

[5]Note that, under (26), $r_\infty = f'(\widetilde{\eta}) > f'(\widehat{\eta}) = f'\left[(f')^{-1}(\rho)\right] = \rho$.

[6]This holds for all $A > 0$ when $m > 2$, while for A large enough if $m = 2$.

where $\tilde{y}(A) = F\left[\tilde{k}(A), A\right] = Af\left[\tilde{k}(A)/A\right]$, with $f(\cdot)$ defined in (10). Note that (29) is the unique dynamic constraint since $\tilde{k} = \tilde{k}(A)$ and thus $\dot{\tilde{k}} = \tilde{k}'(A)\dot{A} = \tilde{k}'(A)[\tilde{y}(A) - \tilde{c}] / \left[\tilde{k}'(A) + \varphi(A)\right]$; therefore, the unique state variable now is A, while, thanks to (22), the unique control variable is $\tilde{c}$.

Thus, the 'regulator' solves

$$\max_{\{\tilde{c}(t)\}} \int_0^\infty u[\tilde{c}(t)] e^{-\rho t} dt \tag{30}$$

$$\text{subject to} \begin{cases} \dot{A}(t) = \{\tilde{y}[A(t)] - \tilde{c}(t)\} / \left\{\tilde{k}'[A(t)] + \varphi[A(t)]\right\} \\ 0 \leq \tilde{c}(t) \leq \tilde{k}[A(t)] + \tilde{y}[A(t)] \\ A(0) = A_0 > 0. \end{cases}$$

The current-value Hamiltonian for problem (30) is

$$\tilde{H}(A, \tilde{c}, \vartheta) = u(\tilde{c}) + \vartheta \frac{\tilde{y}(A) - \tilde{c}}{\tilde{k}'(A) + \varphi(A)}, \tag{31}$$

where ϑ is the costate variable associated with A. Necessary conditions are the following:

$$u'(\tilde{c}) = \frac{\vartheta}{\tilde{k}'(A) + \varphi(A)} \tag{32}$$

$$\dot{\vartheta} = \left\{\rho - \frac{\tilde{y}'(A) - \left[\tilde{k}''(A) + \varphi'(A)\right]\dot{A}}{\tilde{k}'(A) + \varphi(A)}\right\}\vartheta \tag{33}$$

$$\lim_{t\to\infty} \tilde{H}(t) e^{-\rho t} = 0, \tag{34}$$

where $\dot{A}$ in (33) is given by (29).

Since, by (20), $F_A\left[\tilde{k}(A), A\right] = F_k\left[\tilde{k}(A), A\right]\varphi(A)$ along the turnpike and, by (11), $\tilde{r}(A) = F_k\left[\tilde{k}(A), A\right]$, where $\tilde{r}(A)$ is the capital rental rate on the turnpike when the stock of knowledge is A, $\tilde{y}'(A) = F_A\left[\tilde{k}(A), A\right] + F_k\left[\tilde{k}(A), A\right]\tilde{k}'(A) = F_k\left[\tilde{k}(A), A\right]\left[\varphi(A) + \tilde{k}'(A)\right] = \tilde{r}(A)[\varphi(A) + \tilde{k}'(A)\Big]$. Hence, dividing by ϑ, (33) can be rewritten as

$$\frac{\dot{\vartheta}}{\vartheta} = \rho - \tilde{r}(A) + \frac{\left[\tilde{k}''(A) + \varphi'(A)\right]}{\tilde{k}'(A) + \varphi(A)}\dot{A}. \tag{35}$$

By rewriting (32) as $\vartheta = u'(\tilde{c})\left[\tilde{k}'(A) + \varphi(A)\right]$ and taking time derivative we get

$$\dot{\vartheta} = u''(\tilde{c})\dot{\tilde{c}}\left[\tilde{k}'(A) + \varphi(A)\right] + u'(\tilde{c})\left[\tilde{k}''(A) + \varphi'(A)\right]\dot{A},$$

which, divided by $\vartheta = u'(\tilde{c})\left[\tilde{k}'(A) + \varphi(A)\right]$ yields

$$\frac{\dot{\vartheta}}{\vartheta} = \frac{u''(\tilde{c})}{u'(\tilde{c})}\dot{\tilde{c}} + \frac{\left[\tilde{k}''(A) + \varphi'(A)\right]}{\tilde{k}'(A) + \varphi(A)}\dot{A}.$$

Coupling the last equality with (35) and using Assumption A.3, we easily obtain:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{\tilde{r}(A) - \rho}{\sigma} = \frac{f'\left[\tilde{k}(A)/A\right] - \rho}{\sigma}, \tag{36}$$

where in the second equality (11) and (10) have been used.

Therefore, from (29) and (36) we obtain the following system of ODEs defining the optimal dynamics for the state variable $A(t)$ and the control $\tilde{c}(t)$ through time along the turnpike under Assumption A.3:

$$\begin{cases} \dot{A} = \dfrac{\tilde{y}(A) - \tilde{c}}{\tilde{k}'(A) + \varphi(A)} = \dfrac{f\left[\tilde{k}(A)/A\right]A - \tilde{c}}{\tilde{k}'(A) + \varphi(A)} \\ \dot{\tilde{c}} = \dfrac{f'\left[\tilde{k}(A)/A\right] - \rho}{\sigma}\tilde{c}, \end{cases} \tag{37}$$

Proposition 2 (ii) states that in the long run the ratios $\dot{A}/A$ and $\dot{\tilde{c}}/\tilde{c}$ obtained from (37) converge to the balanced growth rate $\gamma = (r_\infty - \rho)/\sigma$, where $r_\infty = \lim_{A\to\infty} f'\left[\tilde{k}(A)/A\right] = \lim_{A\to\infty} f'\left[\tilde{k}_\infty(A)/A\right] = f'(\tilde{\eta})$, with $\tilde{\eta}$ defined in (23).

5 Model specification

We now continue our analysis by suitably restricting the class of models under investigation.

A. 4 *In addition to Assumption A.3 , the followings hold.*

(i) *Only pairs of ideas will be matched together in the recombinant process; i.e.,* $m = 2$.

(ii) *The probability function* $\pi : \mathbb{R}_+ \to [0,1]$ *of the Weitzman's recombinant process*[7] *has the following form:*

$$\pi(x) = \frac{\beta x}{\beta x + 1}, \qquad \beta > 0. \tag{38}$$

(iii) *The production function has the Cobb-Douglas form:*

$$F(k, A) = \theta k^\alpha A^{1-\alpha} = \theta A\left(\frac{k}{A}\right)^\alpha, \qquad \theta > 0,\ 0 < \alpha < 1.$$

Clearly, the function π defined in (38) satisfies Assumption A.1. Parameter β in (38) measures the degree of efficiency of the Weitzman matching process; specifically, the larger β, the higher probability of obtaining a new successful idea out of the same number of (pairwise) matchings of hybrid ideas.

Assumption 4(i) and (ii) allows for a direct computation of the function $\varphi(A)$. Since, when $m = 2$, $C_2'(A) = (2A-1)/2$, and from (38) we get $\pi^{-1}(y) = y/[\beta(1-y)]$, substituting both in (7) yields the following analytical form for the unit cost of knowledge production:

$$\varphi(A) = \frac{2A-1}{\beta(2A-3)} = \frac{1}{\beta}\left(1 + \frac{2}{2A-3}\right). \tag{39}$$

[7]See Section 2.1.

As $\pi'(0) = \beta$, Assumption 4(iii) and (39) yields:

$$\tilde{k}(A) = \frac{\alpha}{1-\alpha}\varphi(A)A = \frac{\alpha}{\beta(1-\alpha)}\left(1 + \frac{2}{2A-3}\right)A \tag{40}$$

$$\tilde{k}_\infty(A) = \frac{\alpha}{\beta(1-\alpha)}(A+1) \qquad \left(\textit{i.e.}, \tilde{\eta} = q = \frac{\alpha}{\beta(1-\alpha)}\right) \tag{41}$$

$$\hat{k}(A) = \left(\frac{\theta\alpha}{\rho}\right)^{1/(1-\alpha)} A \qquad \left(\textit{i.e.}, \hat{\eta} = \left(\frac{\theta\alpha}{\rho}\right)^{1/(1-\alpha)}\right), \tag{42}$$

and the growth condition (26) becomes

$$\rho < \theta\alpha\left[\frac{\beta(1-\alpha)}{\alpha}\right]^{1-\alpha}. \tag{43}$$

5.1 Preliminary features of the policy along the turnpike

It is easily seen from (40) that $\tilde{k}(A)$ diverges to $+\infty$ when A approaches $3/2$ from the right; therefore we must restrict the range for the feasible initial conditions A_0 to the open interval $(3/2, +\infty)$. Since $\tilde{k}(A)$ approaches the asymptotic turnpike $\tilde{k}_\infty(A)$ from above for large A, and $\tilde{k}_\infty(A)$ is increasing, it is easily understood that the graph of $\tilde{k}(A)$ on the whole interval $(3/2, +\infty)$ must be a U-shaped curve. Since the stock of knowledge A cannot be depleted and the economy is bound to follow the optimal (strictly positive) investment in R&D policy $\tilde{J}$ defined in (22), on such graph – *i.e.*, along the turnpike – the stock of knowledge A must grow through time; that is, $\dot{A}(t) > 0$ must hold for all $t \geq 0$. A U-shaped curve for $\tilde{k}(A)$ implies that, while A keeps growing, the optimal amount of capital $\tilde{k}[A(t)]$ must decrease when t is small, and increase for larger t; in other words, $\dot{\tilde{k}}(t) < 0$ for small t, and $\dot{\tilde{k}}(t) > 0$ as t becomes larger, envisaging that in early times it is optimal to take away from the output-producing sector some physical capital and invest it in R&D, in order to allow the stock of knowledge A to take-off.

Having $\dot{A}(t) > 0$ for all $t \geq 0$ has important implications for the study of system (37), as can be easily grasped from equation (29).

Proposition 3 *Under Assumption A.4, the* optimal policy *along the turnpike,* $\tilde{c}(A)$, *must necessarily satisfy*

$$\begin{cases} \tilde{c}(A) > \tilde{y}(A) \ \textit{for}\ 3/2 < A < A^s \\ \tilde{c}(A^s) = \tilde{y}(A^s) \\ \tilde{c}(A) < \tilde{y}(A) \ \textit{for}\ A > A^s, \end{cases} \tag{44}$$

where

$$A^s = 1 + \frac{1}{2}\left(\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right). \tag{45}$$

Moreover, $\tilde{c}'(A^s) < 0$ *in a neighborhood of* A^s.

Proof. by differentiating $\tilde{k}(A)$ in (40) it is easily seen that the denominator of (29), $\tilde{k}'(A) + \varphi(A)$, vanishes on the unique point A^s defined in (45), which belongs to the domain $(3/2, +\infty)$ as $A^s > 3/2$ for all $0 < \alpha < 1$; moreover, $\tilde{k}'(A) + \varphi(A) < 0$ for $3/2 < A < A^s$ and $\tilde{k}'(A) + \varphi(A) > 0$ for $A > A^s$. Therefore, $\dot{A}(t) > 0$ for all $t \geq 0$ in (29) implies (44). Since it can be checked that A^s is also the unique (minimum) stationary point for the optimal output $\tilde{y}(A)$ – *i.e.*, $\tilde{y}'(A^s) = 0$ – and (44) states that the graph of the optimal policy $\tilde{c}(A)$ must intersect the graph of the optimal output $\tilde{y}(A)$ from above on $A = A^s$, $\tilde{c}'(A^s) < 0$ must hold in a neighborhood of A^s. ∎

From the proof we also learn that $\tilde{y}(A)$ has a U-shaped graph similar to that of $\tilde{k}(A)$.

By taking the ratio of the two equations in (37) we get the unique differential equation

$$\frac{\dot{\tilde{c}}}{\dot{A}} = \tilde{c}'(A) = \frac{f'\left[\tilde{k}(A)/A\right] - \rho}{\sigma\left[\tilde{y}(A) - \tilde{c}(A)\right]}\left[\tilde{k}'(A) + \varphi(A)\right]\tilde{c}(A) \tag{46}$$

in the sole variable A characterizing the optimal policy $\tilde{c}(A)$ (see Mulligan and Sala-i-Martin ,1991, and Barro and Sala-i-Martin, 2004, pp. 593-596). As a matter of fact, property (44) actually provides the initial condition $\tilde{c}(A^s) = \tilde{y}(A^s)$ for the ODE (46); hence, by replacing all functions in (46) according to Assumption A.4 – *i.e.*, by using (39) and (40) under condition (43) – $\tilde{c}'(A^s)$ can be easily computed by applying l'Hôpital rule to (46) evaluated at $A = A^s$, and taking the negative solution of the second-order equation thus obtained. With this information at hand, one may try to solve (46) numerically in order to find the optimal policy $\tilde{c}(A)$. We actually tried this approach, but the result was not satisfactory, especially for large A; hence, we chose to rely our analysis on the 'detrended' system that will be discussed in the next section.

Nonetheless, Proposition 3 will prove useful in studying the point corresponding to $(A^s, \tilde{c}(A^s))$ in terms of detrended variables.

5.2 State-like and control-like variables

Since an economy growing along the turnpike $\tilde{k}(A)$ in the long run performs sustained growth, there are no steady states toward which the system may eventually converge. Thus, we first need to transform the state variable A and the control variable $\tilde{c}$ in a state-like variable, μ, and a control-like variable, χ, respectively, so that $\mu(t)$ and $\chi(t)$ converge to some fixed points μ^* and χ^* in the space (μ, χ) as time elapses. Specifically, we choose the following transformations:

$$\mu = \frac{\tilde{k}(A)}{A} = \frac{\alpha}{1-\alpha}\varphi(A) = \frac{\alpha}{\beta(1-\alpha)}\left(1 + \frac{2}{2A-3}\right) \tag{47}$$

$$\chi = \frac{\tilde{c}}{A}, \tag{48}$$

where the second equality in (47) holds by (40), and the third by (39). Hence, A is related to μ as follows:

$$A = \frac{\alpha}{\beta(1-\alpha)\mu - \alpha} + \frac{3}{2}. \tag{49}$$

Similarly, provided that one can compute the 'detrended' optimal policy $\chi(\mu)$, the optimal policy of problem (30) turns out to be

$$\tilde{c}(A) = \chi\left[\frac{\alpha}{1-\alpha}\varphi(A)\right]A. \tag{50}$$

Under Assumption A.4(iii), from (37) we obtain the following ratios:

$$\frac{\dot{A}}{A} = \frac{1}{\tilde{k}'(A) + \varphi(A)}\left\{\theta\left[\frac{\tilde{k}(A)}{A}\right]^{\alpha} - \frac{\tilde{c}}{A}\right\} \tag{51}$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\sigma}\left\{\theta\alpha\left[\frac{\tilde{k}(A)}{A}\right]^{\alpha-1} - \rho\right\}. \tag{52}$$

The growth rate of μ in (47) is given by:

$$\frac{\dot{\mu}}{\mu}=\frac{\dot{\tilde{k}}(A)}{\tilde{k}(A)}-\frac{\dot{A}}{A}=\frac{\tilde{k}'(A)\dot{A}}{\tilde{k}(A)}-\frac{\dot{A}}{A},$$

therefore,

$$\dot{\mu}=\frac{\tilde{k}'(A)\dot{A}}{\tilde{k}(A)}\frac{\tilde{k}(A)}{A}-\frac{\dot{A}}{A}\mu=\left[\tilde{k}'(A)-\mu\right]\frac{\dot{A}}{A}, \tag{53}$$

which, coupled with (51) and using (48), yields

$$\dot{\mu}=\frac{\tilde{k}'(A)-\mu}{\tilde{k}'(A)+\varphi(A)}(\theta\mu^{\alpha}-\chi). \tag{54}$$

Note that, as (39) can be rewritten as $2/(2A-3)=\beta\varphi(A)-1$ and $\varphi'(A)=-4/\left[\beta(2A-3)^2\right]$, φ' turns out to be a function of φ: $\varphi'(A)=-(1/\beta)\left[2/(2A-3)\right]^2=-(1/\beta)\left[\beta\varphi(A)-1\right]^2$; moreover, (39) may be rewritten as $A=1/\left[\beta\varphi(A)-1\right]+3/2$. By differentiating (40) and substituting $\varphi'(A)$ and A, after a fair amount of algebra $\tilde{k}'(A)$ in (54) becomes

$$\begin{aligned}\tilde{k}'(A)&=\frac{\alpha}{1-\alpha}\left[\varphi'(A)A+\varphi(A)\right]\\&=\frac{\alpha}{1-\alpha}\left\{-\frac{1}{\beta}\left[\beta\varphi(A)-1\right]^2\left[\frac{1}{\beta\varphi(A)-1}+\frac{3}{2}\right]+\varphi(A)\right\}\\&=\frac{\alpha}{2\beta(1-\alpha)}\left\{6\beta\varphi(A)-3\beta^2\left[\varphi(A)\right]^2-1\right\}\\&=\frac{\alpha}{2\beta(1-\alpha)}\left[6\beta\left(\frac{1-\alpha}{\alpha}\right)\mu-3\beta^2\left(\frac{1-\alpha}{\alpha}\right)^2\mu^2-1\right],\end{aligned} \tag{55}$$

where in the last line we used (47) to replace $\varphi(A)=\left[(1-\alpha)/\alpha\right]\mu$. We can now rewrite (54) only in terms of variables μ and χ:

$$\dot{\mu}=\left[1-\frac{2\beta(1-\alpha)\mu}{2\beta(1-\alpha)(1+2\alpha)\mu-3\beta^2(1-\alpha)^2\mu^2-\alpha^2}\right](\theta\mu^{\alpha}-\chi). \tag{56}$$

Similarly, using (51), (52) and (47), the growth rate of χ in (48) is given by:

$$\frac{\dot{\chi}}{\chi}=\frac{\dot{\tilde{c}}}{\tilde{c}}-\frac{\dot{A}}{A}=\frac{\theta\alpha\mu^{\alpha-1}-\rho}{\sigma}-\frac{\theta\mu^{\alpha}-\chi}{\tilde{k}'(A)+\varphi(A)},$$

which, by replacing $\tilde{k}'(A)$ as in (55) and $\varphi(A)=\left[(1-\alpha)/\alpha\right]\mu$, yields the following ODE for the control-like variable χ:

$$\dot{\chi}=\left[\frac{\theta\alpha\mu^{\alpha-1}-\rho}{\sigma}-\frac{2\alpha\beta(1-\alpha)(\theta\mu^{\alpha}-\chi)}{2\beta(1-\alpha)(1+2\alpha)\mu-3\beta^2(1-\alpha)^2\mu^2-\alpha^2}\right]\chi. \tag{57}$$

Hence, we must study the following system of ODEs:

$$\begin{cases}\dot{\mu}=\left[1-\dfrac{2\beta(1-\alpha)\mu}{Q(\mu)}\right](\theta\mu^{\alpha}-\chi)\\ \dot{\chi}=\left[\dfrac{\theta\alpha\mu^{\alpha-1}-\rho}{\sigma}-\dfrac{2\alpha\beta(1-\alpha)(\theta\mu^{\alpha}-\chi)}{Q(\mu)}\right]\chi,\end{cases} \tag{58}$$

where

$$Q(\mu)=-3\beta^2(1-\alpha)^2\mu^2+2\beta(1-\alpha)(1+2\alpha)\mu-\alpha^2. \tag{59}$$

5.3 Fixed points and phase diagram

In Section 5.1 we have seen that $A > 3/2$ must hold. Using this information in (47) one immediately obtains the range $[\mu^*, +\infty)$, with

$$\mu^* = \frac{\alpha}{\beta(1-\alpha)}, \tag{60}$$

for the state-like variable μ, with endpoints corresponding to $A \to +\infty$ and $A \to (3/2)^+$ respectively. In other words, μ^* in (60) is the *steady value* for variable μ corresponding to long-run behavior of the economy along the asymptotic turnpike $\tilde{k}_\infty(A)$ [μ^* is the slope of $\tilde{k}_\infty(A)$, as seen in (41)]. The feasible set for the detrended variables (μ, χ) therefore is $S = [\mu^*, +\infty) \times \mathbb{R}_{++}$.

By studying the first equation in (58), two loci in the space S containing the points such that $\dot{\mu} = 0$ are found: the curve

$$\chi = \theta\mu^\alpha \tag{61}$$

and the vertical line $\mu^* \equiv \alpha/[\beta(1-\alpha)]$ defined in (60). Equation (61) vanishes the second factor in the RHS of the first equation in (58), while μ^* is the largest (and the only feasible) solution of

$$Q(\mu) - 2\beta(1-\alpha)\mu = -3\beta^2(1-\alpha)^2\mu^2 + 4\alpha\beta(1-\alpha)\mu - \alpha^2 = 0,$$

which vanishes the first factor in the same equation.

It is clear from the second equation in (58) that all the points (μ, χ) such that $\dot{\chi} = 0$ are on the unique locus

$$\chi = \theta\mu^\alpha - \frac{Q(\mu)}{2\alpha\beta\sigma(1-\alpha)}\left(\theta\alpha\mu^{\alpha-1} - \rho\right). \tag{62}$$

By studying the sign of the function $Q(\mu)$ defined in (59), we find a unique (admissible) root, call it μ^s, solving

$$Q(\mu) = -3\beta^2(1-\alpha)^2\mu^2 + 2\beta(1-\alpha)(1+2\alpha)\mu - \alpha^2 = 0, \tag{63}$$

while $Q(\mu) > 0$ for $\mu^* \leq \mu < \mu^s$ and $Q(\mu) < 0$ for $\mu > \mu^s$. Thus, whether the locus in (62) lies above or below the locus in (61) depends on whether $\mu^* \leq \mu < \mu^s$ or $\mu > \mu^s$, and on the sign of $(\theta\alpha\mu^{\alpha-1} - \rho)$; on $\mu = \mu^s$, however, they intersect, and this yields the *first steady state* of our system: (μ^s, χ^s), with $\chi^s = \theta(\mu^s)^\alpha$.

It turns out that (μ^s, χ^s) corresponds to the point $(A^s, \tilde{c}(A^s))$ for the original dynamic (37) discussed in Proposition 3. To see this, recall that, from (44), $\tilde{c}(A^s) = \tilde{y}(A^s)$ must hold on the critical point A^s defined in (45); by replacing A^s in (47) and (48), we get,

$$\mu^s = \frac{\alpha}{1-\alpha}\varphi(A^s) = \frac{1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}}{3\beta(1-\alpha)} \tag{64}$$

$$\chi^s = \frac{\tilde{y}(A^s)}{A^s} = \theta\left[\frac{\tilde{k}(A^s)}{A^s}\right]^\alpha = \theta(\mu^s)^\alpha, \tag{65}$$

where μ^s in (64) coincides with the largest (and the only admissible) solution of (63).

It is immediately seen that $\mu^* < \mu^s$ for all feasible values of parameters α and β, which means that $Q(\mu^*) > 0$ must hold; moreover, using (60), the necessary condition for growth (43) can be rewritten as $\rho < \theta\alpha(\mu^*)^{\alpha-1}$, that is, $\left[\theta\alpha(\mu^*)^{\alpha-1} - \rho\right] > 0$. Therefore, we can conclude from (62) that the locus $\dot{\chi} = 0$ intersects the vertical line $\mu^* \equiv \alpha/[\beta(1-\alpha)]$ strictly below the locus $\dot{\mu} = 0$ in (61). Since along such line $\dot{\mu} = 0$ as well, we have found the *second steady state* of system (58): (μ^*, χ^*), where χ^* is (62) evaluated at $\mu = \mu^*$, specifically,

$$\chi^* = \theta\left[\frac{\alpha}{\beta(1-\alpha)}\right]^\alpha\left(1 - \frac{1}{\sigma}\right) + \frac{\rho}{\beta\sigma(1-\alpha)}. \tag{66}$$

As $\theta\mu^\alpha$ in (61) is increasing and $\chi^* < \theta(\mu^*)^\alpha$, it follows that (μ^*, χ^*) lies south-west of (μ^s, χ^s). We shall see in short that (μ^*, χ^*) is the saddle-path stable steady state to which system (58) converges in the long-run. More precisely, χ^* in (66) corresponds to the slope of the optimal consumption $\tilde{c}(A)$ steadily growing at the constant rate γ along the asymptotic turnpike $\tilde{k}_\infty(A)$ in the original model.

As the necessary condition for growth (43) states that $\rho < \theta\alpha(\mu^*)^{\alpha-1}$ must always be satisfied and, as $0 < \alpha < 1$, $\theta\alpha\mu^{\alpha-1}$ is a decreasing function of μ, it follows that there must be a unique value $\hat{\mu} > \mu^*$ such that $\theta\alpha(\hat{\mu})^{\alpha-1} = \rho$, that is, $\left[\theta\alpha(\hat{\mu})^{\alpha-1} - \rho\right] = 0$. It is clear from the last factor in the second term in the RHS of (62) that the two loci $\dot{\chi} = 0$ in (62) and $\dot{\mu} = 0$ in (61) must intersect in $\mu = \hat{\mu}$, hence $(\hat{\mu}, \hat{\chi})$, with

$$\hat{\mu} = \left(\frac{\theta\alpha}{\rho}\right)^{\frac{1}{1-\alpha}} \tag{67}$$

$$\hat{\chi} = \theta\left(\frac{\theta\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}}, \tag{68}$$

is the *third* (and last) *steady state* associated to (58). It is worth noting that $\hat{\mu}$ in (67) corresponds to the (unique) value $\hat{A}$ at which the turnpike $\tilde{k}(A)$ intersects the stagnation line $\hat{k}(A)$ in the original model, as it becomes clear from (42). By equating (40) and (42) [or by substituting $\hat{\mu}$ as in (67) into (49)], $\hat{A}$ turns out to be

$$\hat{A} = \frac{\alpha}{\beta(1-\alpha)(\theta\alpha/\rho)^{\frac{1}{1-\alpha}} - \alpha} + \frac{3}{2}, \tag{69}$$

which in turn, if replaced in (50) and using $\hat{\chi}$ as in (68), yields the value of the optimal policy at the intersection point $\hat{A}$, $\tilde{c}\left(\hat{A}\right) = \hat{\chi}\hat{A}$, of the original model.

The position of the last steady state, $(\hat{\mu}, \hat{\chi})$, depends on how large the discount factor ρ is with respect to the parameters α, θ and β. Since $\mu^* < \mu^s$ implies $\theta\alpha(\mu^s)^{\alpha-1} < \theta\alpha(\mu^*)^{\alpha-1}$, three scenarios may occur, all satisfying condition (43).

1. If $\rho < \theta\alpha(\mu^s)^{\alpha-1}$, then $\mu^s < (\theta\alpha/\rho)^{\frac{1}{1-\alpha}} = \hat{\mu}$; hence, $(\hat{\mu}, \hat{\chi})$ lies north-east of (μ^s, χ^s) [as $\theta\mu^\alpha$ in (61) is increasing].

2. If $\rho = \theta\alpha(\mu^s)^{\alpha-1}$, then $\mu^s = (\theta\alpha/\rho)^{\frac{1}{1-\alpha}} = \hat{\mu}$, and thus the two steady states collapse: $(\hat{\mu}, \hat{\chi}) = (\mu^s, \chi^s)$.

3. Finally, if $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$, then $\mu^* < (\theta\alpha/\rho)^{\frac{1}{1-\alpha}} = \hat{\mu} < \mu^s$; therefore $(\hat{\mu}, \hat{\chi})$ lies north-east of (μ^*, χ^*) and south-west of (μ^s, χ^s).

In this paper we shall focus on the third case, which corresponds to a scenario in which the critical point A^s defined in (45) lies on the left of the intersection point $\hat{A}$ defined in (69) on which the turnpike $\tilde{k}(A)$ intersects (from above) the stagnation line $\hat{k}(A)$.

Proposition 4 *Under Assumption A.4 and provided that* $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$ *holds, the two fixed points* (μ^*, χ^*) *and* $(\hat{\mu}, \hat{\chi})$ *defined above can be classified as follows in terms of their stability properties.*

1. $(\mu^*, \chi^*) = (\alpha/[\beta(1-\alpha)], \theta\{\alpha/[\beta(1-\alpha)]\}^\alpha(1-1/\sigma 1) + \rho[\beta\sigma(1-\alpha)])$ *is* saddle-path stable, *with the stable arm converging to it from north-east whenever the initial values* $(\mu(t_0), \chi(t_0))$ *are suitably chosen.*

2. $(\hat{\mu}, \hat{\chi}) = \left((\theta\alpha/\rho)^{\frac{1}{1-\alpha}}, \theta(\theta\alpha/\rho)^{\frac{\alpha}{1-\alpha}}\right)$ *is an unstable clockwise-rotating spiral.*

The third point, (μ^s, χ^s), cannot be classified analytically.

Proof. It is immediately seen that the Jacobian matrix of (58) evaluated at $(\mu^s, \chi^s) = \left(\left[1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right] / \left[3\beta\left(1-\alpha\right)\right], \theta\left\{\left[1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right] / \left[3\beta\left(1-\alpha\right)\right]\right\}^{\alpha}\right)$ is undefined, as some of its elements diverge either to $-\infty$ or to $+\infty$ as (μ, χ) approaches (μ^s, χ^s), the sign of infinity depending on the direction along which $(\mu, \chi) \to (\mu^s, \chi^s)$. We thus focus our attention on the other two steady states.

Above the locus $\dot{\mu} = 0$ (61), $\chi = \theta\mu^\alpha$, the term $(\theta\mu^\alpha - \chi)$ in the first equation of (58) is negative, while it is positive below the same locus. $Q(\mu)$ defined in (59) is such that $Q(\mu) > 0$ for $\mu^* < \mu < \mu^s$, while $Q(\mu) < 0$ for $\mu > \mu^s$; therefore, $[1 - 2\beta(1-\alpha)\mu/Q(\mu)]$ turns out to be negative for $\mu^* < \mu < \mu^s$, while it is positive for $\mu > \mu^s$. Hence: if $\mu^* < \mu < \mu^s$, $\dot{\mu} > 0$ above the locus (59) and $\dot{\mu} < 0$ below the same locus; while, if $\mu > \mu^s$, $\dot{\mu} < 0$ above the locus (59) and $\dot{\mu} > 0$ below the same locus.

Since $\chi > 0$, the sign of $\dot{\chi}$ in the second equation of (58) depends on the sign of the term in square brackets in the RHS. As $Q(\mu) > 0$ for $\mu^* < \mu < \mu^s$ and $Q(\mu) < 0$ for $\mu > \mu^s$, it turns out that in the first case such term is positive above the locus (62), $\chi = \theta\mu^\alpha - Q(\mu)\left(\theta\alpha\mu^{\alpha-1} - \rho\right) / \left[2\alpha\beta\sigma(1-\alpha)\right]$, while it is negative below the same locus; the converse holds for $\mu > \mu^s$. Thus, when $\mu^* < \mu < \mu^s$, $\dot{\chi} > 0$ above the locus (62), while $\dot{\chi} < 0$ below the same locus; conversely, if $\mu > \mu^s$, $\dot{\chi} < 0$ above the locus (62), while $\dot{\chi} > 0$ below the same locus.

The analysis above is sufficient to trace out the whole phase diagram for the case $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$, that is, when $\mu^* < \hat{\mu} < \mu^s$, which is fully reported in Figure 1. It is clearly seen that the fixed point (μ^*, χ^*), whose coordinates are defined in (60) and (66) respectively, is *saddle-path stable*; it can be guessed that its stable arm is increasing and lying below the locus (62) on the interval $[\mu^*, \mu^s)$. To check its saddle-path stability, consider the Jacobian of (58) evaluated at (μ^*, χ^*):

$$J(\mu^*, \chi^*) = \begin{bmatrix} \left[\rho - \beta\theta(1-\alpha)(\mu^*)^\alpha\right]/\sigma & 0 \\ -\frac{1-\alpha}{\alpha\sigma^2}\left[c_1(\mu^*)^{2\alpha} + c_2(\mu^*)^\alpha + \rho^2\right] & \left[\rho + \beta\theta(1-\alpha)(\sigma-1)(\mu^*)^\alpha\right]/\sigma \end{bmatrix}, \tag{70}$$

where $c_1 = (\beta\theta)^2\alpha\sigma(1-\alpha)(\sigma-1)$ and $c_2 = \beta\theta\rho(\alpha+\sigma-1)$. Note that, by (43), the term on the top left is negative; while the term on the bottom right is clearly positive. Hence, $\det[J(\mu^*, \chi^*)] < 0$, establishing that (μ^*, χ^*) is a saddle.

As the fixed point (μ^*, χ^*) lies strictly below the locus (61) and the unique intersection point between the loci (62) and (61) is the fixed point $(\hat{\mu}, \hat{\chi})$, whose coordinates are defined in (67) and (68) respectively, it must be the case that (62) crosses (61) from below on $(\hat{\mu}, \hat{\chi})$. Therefore, $(\hat{\mu}, \hat{\chi})$ must be a clockwise rotating *spiral* and the eigenvalues of the Jacobian of (58) evaluated at $(\hat{\mu}, \hat{\chi})$ are complex. Thus, to establish instability we need to show that their real part is positive, or, equivalently, that $\mathrm{tr}[J(\hat{\mu}, \hat{\chi})] > 0$. The Jacobian is

$$J(\hat{\mu}, \hat{\chi}) = \frac{1}{Q(\hat{\mu})}\begin{bmatrix} \left[Q(\hat{\mu}) - 2\beta(1-\alpha)\hat{\mu}\right]\rho & 2\beta(1-\alpha)\hat{\mu} - Q(\hat{\mu}) \\ -\frac{1-\alpha}{\sigma}\left[\rho Q(\hat{\mu}) + 2\sigma\alpha^2\beta\theta(\hat{\mu})^\alpha + \rho^2\right]\theta(\hat{\mu})^{\alpha-1} & 2\alpha\beta(1-\alpha)\theta(\hat{\mu})^\alpha \end{bmatrix},$$

with $Q(\hat{\mu}) > 0$, as $\mu^* < \hat{\mu} < \mu^s$. Hence,

$$\begin{aligned} \mathrm{tr}[J(\hat{\mu}, \hat{\chi})] &= \rho Q(\hat{\mu}) - 2\beta(1-\alpha)\hat{\mu}\rho + 2\alpha\beta(1-\alpha)\theta(\hat{\mu})^\alpha \\ &= \rho Q(\hat{\mu}) + 2\beta(1-\alpha)\hat{\mu}\left[\alpha\theta(\hat{\mu})^{\alpha-1} - \rho\right] \\ &= \rho Q(\hat{\mu}) > 0 \end{aligned}$$

where the last equality holds as $\left[\alpha\theta(\hat{\mu})^{\alpha-1} - \rho\right] = 0$ on the intersection between the loci (61) and (62) on $(\hat{\mu}, \hat{\chi})$. This completes the proof. ■

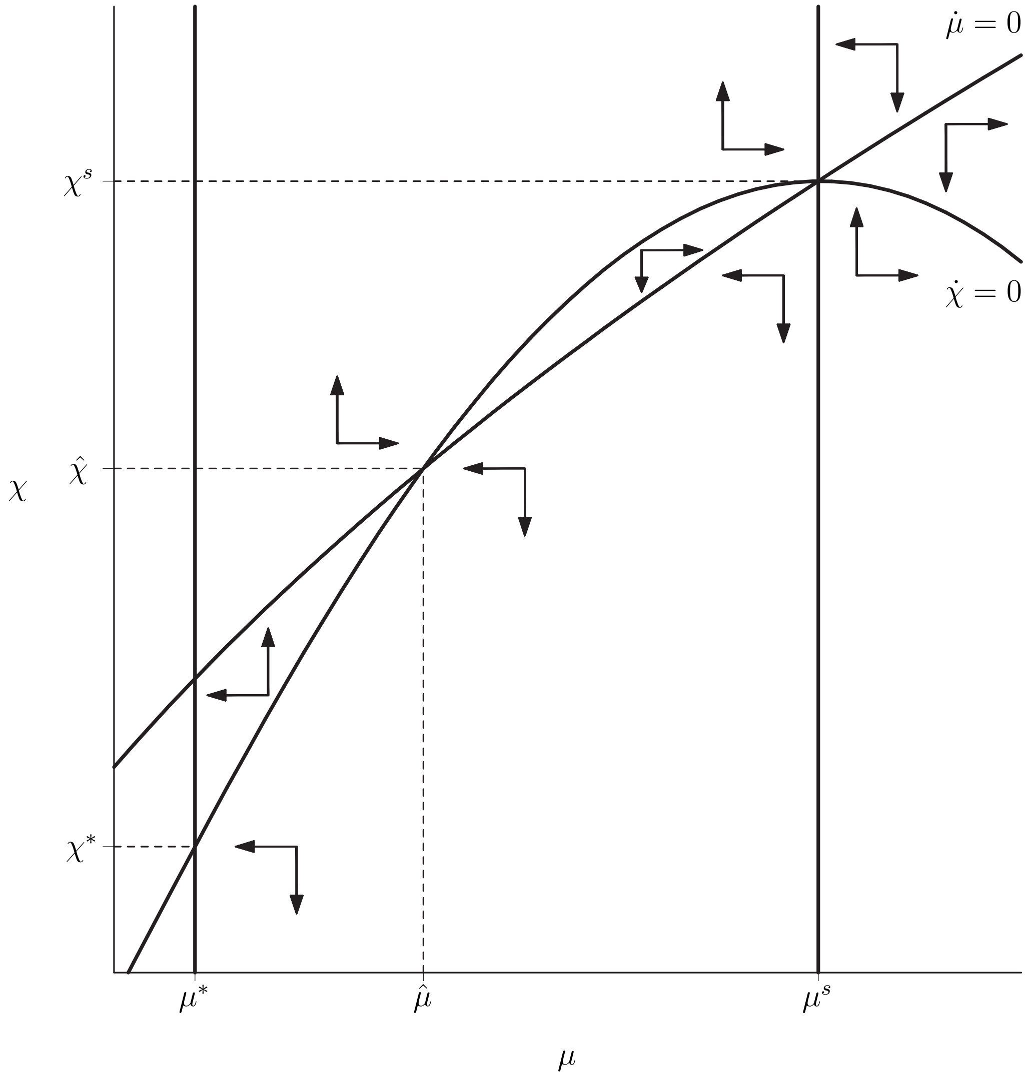

FIGURE 1: phase diagram of system (58) when $\theta\alpha (\mu^s)^{\alpha-1} < \rho < \theta\alpha (\mu^*)^{\alpha-1}$.

We have seen in Section 3 that the turnpike $\tilde{k}(A)$ lies always above the asymptotic turnpike, $\tilde{k}(A) > \tilde{k}_\infty (A)$ for all A (and thus for all instants t); this is consistent with the fact that $\mu(t) > \mu^*$ must hold for all t and thus, the stable trajectory must approach the fixed point (μ^*, χ^*) from the right. We denote by $\chi(\mu)$ the stable trajectory converging to (μ^*, χ^*); that is, $\chi(\mu)$ is the *optimal policy expressed in terms of state-like and control-like variables* according to (58). Its slope on the fixed point (μ^*, χ^*) is given by the slope of the eigenvector associated to the negative eigenvalue of the Jacobian $J(\mu^*, \chi^*)$ defined in (70) (see Barro and Sala-i-Martin, 2004, p. 596). The eigenvalues of $J(\mu^*, \chi^*)$ are the elements on its diagonal: $[\rho - \beta\theta(1-\alpha)(\mu^*)^\alpha]/\sigma$ and $[\rho + \beta\theta(1-\alpha)(\sigma-1)(\mu^*)^\alpha]/\sigma$ respectively, where the first one is negative; its associated eigenvector has slope given by:

$$\begin{aligned}\chi'(\mu^*) &= -\frac{-(1-\alpha)\left[c_1 (\mu^*)^{2\alpha} + c_2 (\mu^*)^\alpha + \rho^2\right]/(\alpha\sigma^2)}{[\rho + \beta\theta(1-\alpha)(\sigma-1)(\mu^*)^\alpha]/\sigma - [\rho - \beta\theta(1-\alpha)(\mu^*)^\alpha]/\sigma} \\ &= \frac{(\beta\theta)^2 \alpha\sigma(1-\alpha)(\sigma-1)(\mu^*)^{2\alpha} + \beta\theta\rho(\alpha+\sigma-1)(\mu^*)^\alpha + \rho^2}{\alpha\sigma^2\beta\theta(\mu^*)^\alpha} \\ &= \frac{\beta\theta\alpha\sigma(1-\alpha)(\sigma-1)(\mu^*)^{2\alpha} + \rho(\alpha+\sigma-1)(\mu^*)^\alpha + \rho^2}{\alpha\sigma^2(\mu^*)^\alpha}, \end{aligned} \tag{71}$$

which is clearly positive.

Therefore, the optimal trajectory approaches the fixed point from north-east in a (right) neighborhood of μ^*: along the transitional turnpike, $\tilde{k}(A)$, both ratios $\tilde{k}(A)/A$ and $\tilde{c}/A$ must decline in time when they are approaching the asymptotic turnpike $\tilde{k}_\infty(A)$. It is clear, however, from the phase diagram that, as the optimal trajectory $\chi(\mu)$ must lie below the locus $\dot{\chi} = 0$ defined in (62) for $\mu > \mu^*$, such slope must be less than the (positive) slope of the locus (62), which, after some tedious algebra, can be computed to be: $(1-\alpha)\left[\alpha\beta\theta\sigma(\mu^*)^\alpha + \rho\right]/(\alpha\sigma) > 0$.

Under the assumption that $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$, $\mu^* < \hat{\mu} < \mu^s$ holds true; it follows that, by translating $\hat{\mu}$ into $\hat{A}$ [see (69)] through (49) in the original model, the intersection point between the turnpike $\tilde{k}(A)$ and the stagnation line $\hat{k}(A)$ lies on the right of the singular point A^s defined in (45): $A^s < \hat{A}$. Therefore, condition (44) of Proposition 3 implies that $\tilde{c}\left(\hat{A}\right) < \tilde{y}\left(\hat{A}\right)$, which is equivalent to $\chi(\mu) < \theta(\hat{\mu})^\alpha = \hat{\chi}$. Such inequality states that the optimal trajectory $\chi(\mu)$ cannot intersect the (unstable) steady state $(\hat{\mu}, \hat{\chi})$; as a matter of fact, it keeps well below it, and thus $(\hat{\mu}, \hat{\chi})$ happens to be harmless for our analysis, at least for the case[8] $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$.

Conversely, the steady state left out, (μ^s, χ^s), is the most problematic, as, on one hand there is no way of studying its stability analytically, but on the other hand, since condition (44) of Proposition 3 states that $\tilde{c}(A^s) = \tilde{y}(A^s)$, it turns out that $\chi = \theta(\mu^s)^\alpha = \chi^s$ holds true, implying that the optimal policy $\chi(\mu)$ actually must cross it. Hence, we opted for a qualitative approach based on information gathered on a neighborhood of (μ^s, χ^s). Condition (44) of Proposition 3 outside the singular point A^s translates into

$$\begin{cases} \chi(\mu) < \theta(\mu)^\alpha \text{ for } \mu^* < \mu < \mu^s \\ \chi(\mu) > \theta(\mu)^\alpha \text{ for } \mu > \mu^s, \end{cases} \tag{72}$$

which, in turn, means that the optimal policy must lie below the locus $\dot{\mu} = 0$ defined in (61), $\chi = \theta\mu^\alpha$, when $\mu^* < \mu < \mu^s$ and above the same locus for $\mu > \mu^s$. Moreover, a close inspection of a neighborhood of (μ^s, χ^s) in Figure 1 shows that such point is attractive on the area above the locus $\dot{\mu} = 0$ in (61), that is, above $\chi = \theta\mu^\alpha$, and on the right of the line $\mu \equiv \mu^s$, while it is repulsive on the area below the same locus and on the left of the line $\mu \equiv \mu^s$. As $\theta\mu^\alpha$ is increasing, these considerations suggest that the optimal policy $\chi(\mu)$ must be increasing around (μ^s, χ^s) and the optimal trajectory $(\mu(t), \chi(t))$ must cross the singular point (μ^s, χ^s) from north-east to south-west as time elapses.

5.4 Time elimination, policy function and initial conditions

In order to study the policy function $\chi(\mu)$ expressed in terms of control-like and state-like variables χ and μ along the transitory turnpike $\tilde{k}(A)$ we apply the technique developed by Mulligan and Sala-i-Martin (1991). Thus, we write the unique differential equation given by the ratio between the equations in (58):

$$\chi'(\mu) = \frac{\dot{\chi}}{\dot{\mu}} = \frac{\left[\left(\alpha\theta\mu^{\alpha-1} - \rho\right)/\sigma\right] Q(\mu) - 2\alpha\beta(1-\alpha)\left[\theta\mu^\alpha - \chi(\mu)\right]}{\left[Q(\mu) - 2\beta(1-\alpha)\mu\right]\left[\theta\mu^\alpha - \chi(\mu)\right]}\chi(\mu), \tag{73}$$

where $Q(\mu)$ is defined in (59).

The natural choice for the initial condition of (73) is its value on the saddle-path stable steady state: (μ^*, χ^*), whose coordinates are defined in (60) and (66) respectively; moreover, the value of $\chi'(\mu^*)$ given by (71) will be used to let the numerical algorithm choose the direction along the stable arm

[8]The same situation occurs when $\rho < \theta\alpha(\mu^s)^{\alpha-1}$, in which case $\tilde{c}\left(\hat{A}\right) > \tilde{y}\left(\hat{A}\right)$, and thus $\chi(\mu) > \theta(\hat{\mu})^\alpha = \hat{\chi}$. Only when $\rho = \theta\alpha(\mu^s)^{\alpha-1}$, and the two points $\hat{A}$ and A^s collapse, the optimal trajectory necessarily must cross the (unstable) steady state $(\hat{\mu}, \hat{\chi})$; in this case, however, the point $(\hat{\mu}, \hat{\chi}) = (\mu^s, \chi^s)$ inherits the peculiar singularity properties of (μ^s, χ^s), thus becoming a "supersingular" point to be handled with circumspection.

outside the point (μ^*, χ^*). The previous analysis, however, has endowed us with another reference point, the singular point (μ^s, χ^s) – whose coordinates are defined in (64) and (65) respectively – which may be exploited to check whether the optimal trajectory computed from the steady state (μ^*, χ^*) actually crosses such point as well.

Even if, as we have seen in the previous section, the Jacobian of (58) evaluated on (μ^s, χ^s) is intractable, we are able to compute the slope of the policy at $\mu = \mu^s$ by applying l'Hôpital rule to the RHS of (73) evaluated at $\mu = \mu^s$. By differentiating both the numerator and the denominator in the RHS of (73), by taking into account that $Q(\mu^s) = 0$ and $[\theta (\mu^s)^\alpha - \chi(\mu^s)] = 0$, and by substituting into (73) we obtain the following quadratic equation in $\chi'(\mu^s)$:

$$\begin{aligned} &2\beta\sigma(1-\alpha)(\mu^s)[\chi'(\mu^s)]^2 - 4\alpha\beta\sigma(1-\alpha)(\chi^s)[\chi'(\mu^s)] \\ &-\left\{\left[\alpha\theta(\mu^s)^{\alpha-1} - \rho\right]Q'(\mu^s) - 2\alpha^2\beta\sigma\theta(1-\alpha)(\mu^s)^{\alpha-1}\right\}(\chi^s) = 0. \end{aligned} \tag{74}$$

Substituting $\mu^s = \left[1 + 2\alpha + \sqrt{1 + 4\alpha + \alpha^2}\right] / [3\beta(1-\alpha)]$, $\chi^s = \theta(\mu^s)^\alpha$ and $Q'(\mu^s) = -2\beta(1-\alpha)$ $\sqrt{1 + 4\alpha + \alpha^2}$ into (74) two positive real solutions appear, the largest being larger than the slope of the locus $\dot{\mu} = 0$ defined in (61), $\theta\alpha(\mu^s)^{\alpha-1}$. However, this happens only under the assumption that $\theta\alpha(\mu^s)^{\alpha-1} < \rho < \theta\alpha(\mu^*)^{\alpha-1}$; this is why we chose to confine our numerical approach to such scenario.

With all the information gathered so far, we are ready to solve numerically ODE (73) and thus find the (numeric approximation of the) optimal policy $\chi(\mu)$.

6 Numeric simulation of the optimal policy

After several attempts by means of the *Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant* method (see, *e.g.*, Shampine and Corless, 2000) applied to ODE (73) and implemented through Maple 12.02, we eventually were able to find satisfactory result only for single sets of parameters values. Specifically, we chose values for parameters α, ρ, σ and θ which seem reasonable and are often assumed in the macroeconomic literature (see, *e.g.*, Mulligan and Sala-i-Martin, 1993): $\alpha = 0.5$, $\rho = 0.04$ and $\theta = \sigma = 1$. By assuming the same output elasticity, $\alpha = 0.5$, for both physical capital k and stock of knowledge A in the Cobb-Douglas technology we opted for the simplest and most clear-cut case, while the choice of $\theta = 1$ is motivated again by simplicity and the fact that we are not interested in the total factor productivity. The value $\sigma = 1$ for the reciprocal of the intertemporal elasticity of substitution implies logarithmic instantaneous utility. Given the parameters' values above, we shall consider values for the parameter β (the efficiency parameter in the Weitzman process of matching pairs of seed ideas) satisfying the necessary growth condition (43); that is, such that $\beta > 0.0064$.

We planned to exploit both the steady state (μ^*, χ^*) and the singular point (μ^s, χ^s) discussed in the previous section [see (60), (66), (64) and (65) respectively] as initial conditions in order to trace out two separate trajectories for the solution of the same ODE (73) through Maple 12.02 implementation.[9] Under the choice of parameters' values discussed above, it turned out that such two trajectories perfectly match on most of the interval $[\mu^*, \mu^s]$ only for a unique (feasible) value of the technological parameter β: specifically, $\beta = 0.0124$. Therefore, we shall approximate the optimal policy $\chi(\mu)$ by numerically solving ODE (73) for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

By using this set of parameters' values in (64), it turns out that $\mu^s = 204.4503$, which implies that $\rho = 0.04 > 0.035 = \theta\alpha(\mu^s)^{\alpha-1}$; therefore our example satisfies condition $\rho > \theta\alpha(\mu^s)^{\alpha-1}$,

[9]Apparently, the improvement of the algorithm '*dsolve/numeric*' in the recent update from version 12 to version 12.02 of Maple has been crucial: we would not have been able to evaluate our trajectory for $\mu > \mu^s$ through the older release 12.

corresponding to the third scenario discussed in Section 5.3, in which the critical point A^s defined in (45) lies on the left of the intersection point $\hat{A}$ defined in (69) on which the turnpike $\tilde{k}(A)$ intersects (from above) the stagnation line $\hat{k}(A)$. Figure 2 portraits the turnpike $\tilde{k}(A)$, the asymptotic turnpike $\tilde{k}_\infty(A)$ and the stagnation line $\hat{k}(A)$ as defined in (40), (41) and (42) respectively, for our choice of parameters' values; as expected, $A^s = 2.1514 < \hat{A} = 2.567$.

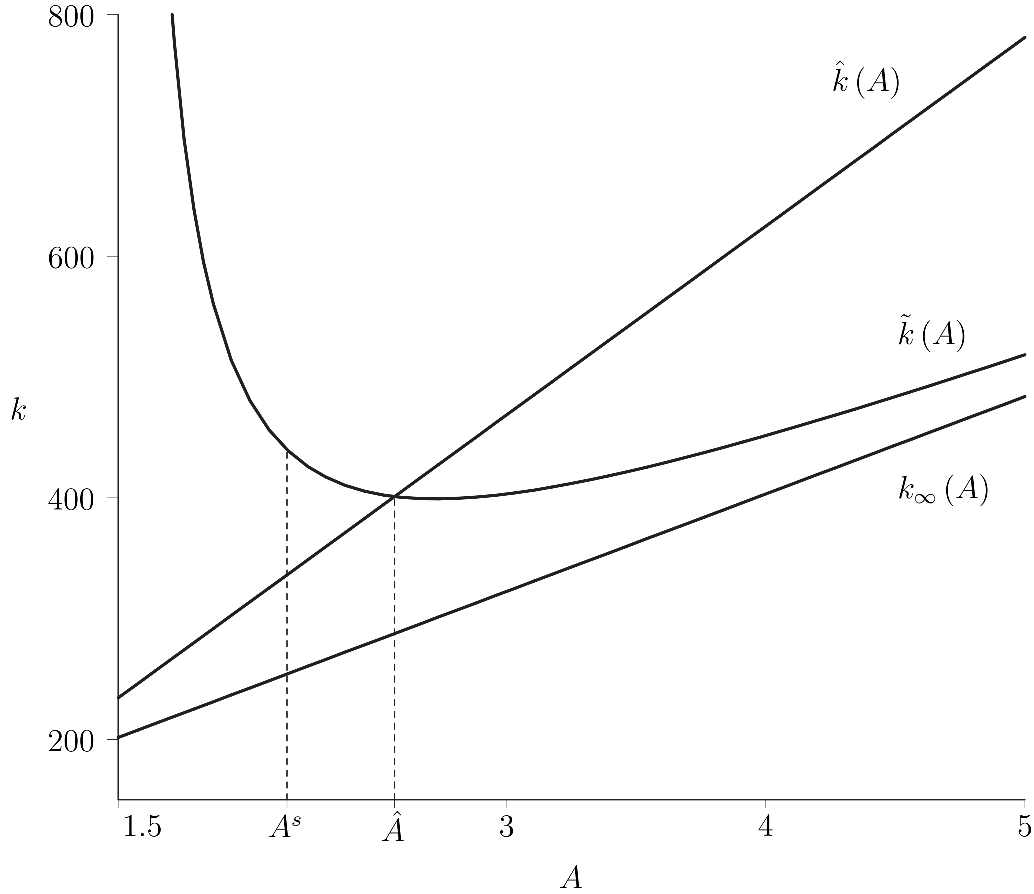

FIGURE 2: the turnpike $\tilde{k}(A)$, the asymptotic turnpike $k_\infty(A)$ and the stagnation line $\hat{k}(A)$ for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

As far as the long-run behavior of the economy is concerned, by (41) $\tilde{\eta} = q = \alpha/[\beta(1-\alpha)] = 80.6452$; thus, the asymptotic turnpike is defined by $\tilde{k}_\infty(A) = 80.6452(A+1)$. In view of Proposition 2 (ii), the long-run capital rental rate is $r_\infty = f'(\tilde{\eta}) = 0.0557$, the long-run common constant growth rate is, according to (27), $\gamma = 0.0157$, while the long-run income shares devoted to investments in knowledge and capital are the same and, according to (28), given by $s_\infty = s_\infty^k = 0.1408$.

The critical values defining the steady states in the phase diagram (see Figure 1) are $(\mu^*, \chi^*) = (80.6452, 6.4516)$, $(\hat{\mu}, \hat{\chi}) = (156.25, 12.5)$ and $(\mu^s, \chi^s) = (204.4503, 14.2986)$.

We now proceed to the numeric computation of two separate solutions of the same ODE (73) through Maple 12.02: the first uses the steady state $(\mu^*, \chi^*) = (80.6452, 6.4516)$ as initial condition and condition (71), $\chi'(\mu^*) = 0.0687$, for the selection of the stable arm, and will be labelled $\chi^*(\mu)$; while the second has the singular point $(\mu^s, \chi^s) = (204.4503, 14.2986)$ as initial condition and has slope given by the larger solution of (74) on $\mu = \mu^s$, *i.e.*, $\chi'(\mu^s) = 0.0602$, and will be denoted by $\chi^s(\mu)$. The same two loci $\dot{\mu} = 0$ and $\dot{\chi} = 0$ of Figure 1 are the two slim black curves reported in Figure 3, while the thick curves, the black one and the grey one, represent trajectory $\chi^*(\mu)$ and trajectory $\chi^s(\mu)$ respectively.

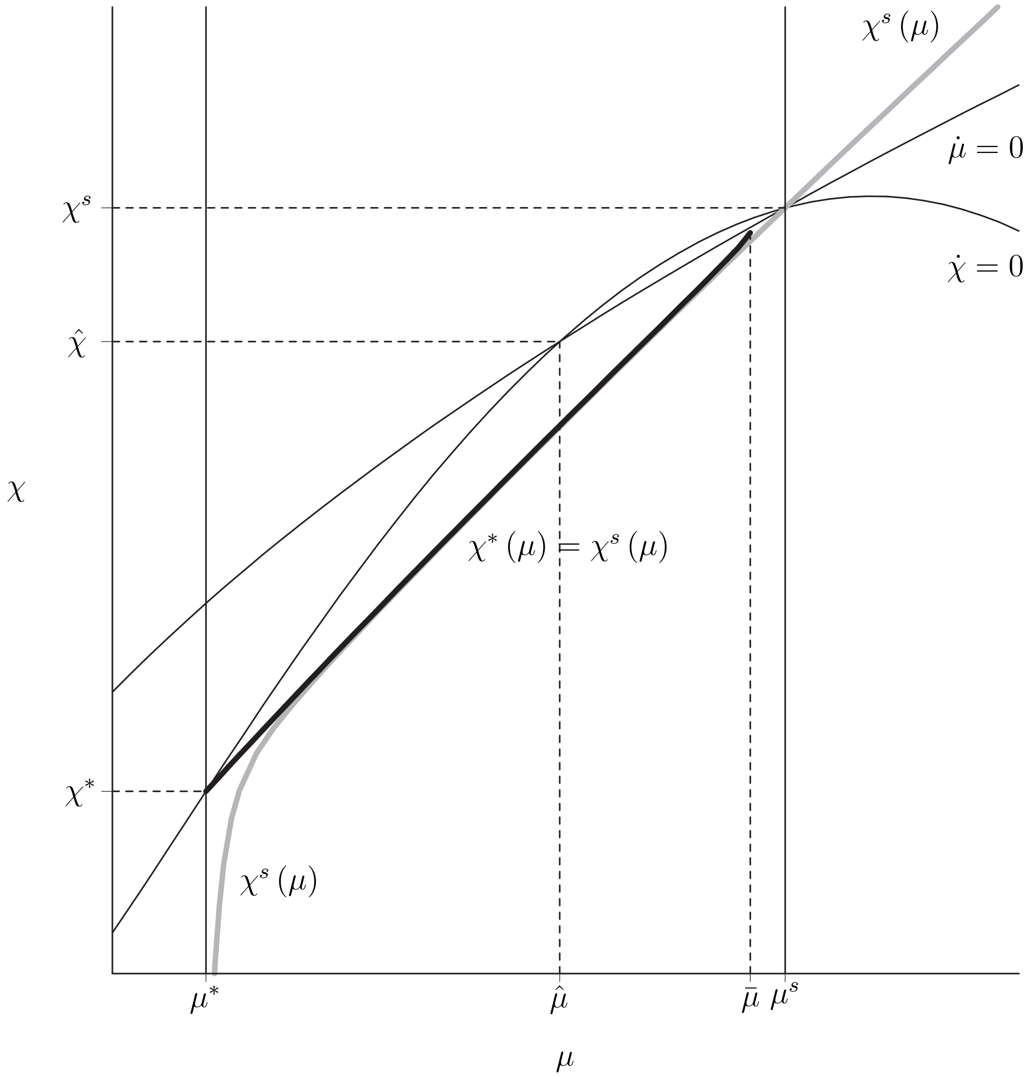

FIGURE 3: the two locuses $\dot{\mu}=0$ and $\dot{\chi}=0$ (the two slim black curves) and the trajectories $\chi^*(\mu)$ and $\chi^s(\mu)$ (the black and the grey thick curves respectively) for $\alpha=0.5$, $\rho=0.04$, $\theta=\sigma=1$ and $\beta=0.0124$.

Even for our peculiar choice of parameters' values the Maple 12.02 algorithm is capable of computing the solution $\chi^*(\mu)$ – starting from the initial condition (μ^*,χ^*) – only up to a point: it actually stops at $\bar{\mu}\simeq 197<204.4503=\mu^s$, thus falling short of the singular point, (μ^s,χ^s). Similarly, as it is clear from Figure 3, the other trajectory, $\chi^s(\mu)$ – using (μ^s,χ^s) as initial conditions – heavily underestimates the policy for values of μ approaching μ^* (*i.e.*, far away from μ^s). The two trajectories, however, perfectly match on most of the (central part of the) interval $[\mu^*,\mu^s]$, thus suggesting that the numeric approach actually works satisfactorily for these values of the parameters. In order to construct our estimation of the whole optimal policy $\chi(\mu)$, for all $\mu\geq\mu^*$, we shall use trajectory $\chi^*(\mu)$ for values of μ close to μ^*, and trajectory $\chi^s(\mu)$ for values of μ in a neighborhood of to μ^s. Specifically, since, as from Figure 3 it is clear that the value $\hat{\mu}$ lies in the part of $[\mu^*,\mu^s]$ on which $\chi^*(\mu)=\chi^s(\mu)$, we shall define the policy by joining the two trajectories on such point, that is, as the following piecewise function:

$$\chi(\mu)=\begin{cases}\chi^*(\mu) & \text{for } \mu^*\leq\mu\leq\hat{\mu}\\ \chi^s(\mu) & \text{for } \mu\geq\hat{\mu}.\end{cases} \tag{75}$$

Surprisingly, already by choosing $\beta=0.0123$, or $\beta=0.0125$ the two curves $\chi^*(\mu)$ and $\chi^s(\mu)$

in Figure 3 split apart while, at the same time, the range of values on which the numeric algorithm is able to provide a solution starts to shrink dramatically; for this reason we take as reliable only the solution obtained for $\beta = 0.0124$ and portrayed in Figure 3.

Remark 1 *We tried different values for the parameters α, ρ, σ and θ; for all feasible set of values for such parameters we found a scenario similar to that described above, at least under condition $\theta\alpha\left(\mu^s\right)^{\alpha-1} < \rho < \theta\alpha\left(\mu^*\right)^{\alpha-1}$: only for one specific value of parameter β – related to the choice of α, ρ, σ and θ – the two numerical solutions of the policy $\chi\left(\mu\right)$ in (75) – $\chi^*\left(\mu\right)$ with initial condition $\left(\mu^*, \chi^*\right)$ and $\chi^s\left(\mu\right)$ with initial condition $\left(\mu^s, \chi^s\right)$ – turned out to match perfectly on a large part of the interval $\left[\mu^*, \mu^s\right]$. We conclude, thus, that the numeric approach works satisfactory only on a manifold of dimension one in the parameters' space.*

7 Discussion

7.1 Time-path trajectories of the detrended variables

To obtain the time-path trajectory of the detrended variable μ we substitute the optimal policy $\chi\left(\mu\right)$ as obtained in (75) into the first equation of system (58), yielding the following ODE with respect to time:

$$\dot{\mu}\left(t\right) = \left\{1 - \frac{2\beta\left(1-\alpha\right)\mu\left(t\right)}{Q\left[\mu\left(t\right)\right]}\right\}\left\{\theta\left[\mu\left(t\right)\right]^{\alpha} - \chi\left[\mu\left(t\right)\right]\right\}, \tag{76}$$

where $Q\left(\cdot\right)$ is defined in (59). Again (76) can be numerically solved in the same manner as we did for ODE (73). Since the policy $\chi\left(\mu\right)$ in (75) is defined piecewise, we first need to choose a date $t = \hat{t} > 0$ on which the trajectory assumes the (common) value $\hat{\mu} = 156.25$; then, in order to find the initial value $\mu_0 = \mu\left(0\right)$ for μ in $t = 0$, we solve (76) with $\chi\left[\mu\left(t\right)\right] = \chi^s\left[\mu\left(t\right)\right]$ and with initial condition $\mu\left(\hat{t}\right) = \hat{\mu}$: μ_0 is the value of the numeric solution just computed in $t = 0$. Note that by choosing different values of $\hat{t}$ we can consider any initial value $\mu_0 = \mu\left(0\right) > \hat{\mu}$.

In our example we assume that $\hat{t} = 36$, corresponding to the initial condition $\mu_0 = 251.977$ in $t = 0$. Then, according to (75), we define the time-path trajectory $\mu\left(t\right)$ as the numeric solution of (76) with $\chi\left[\mu\left(t\right)\right] = \chi^s\left[\mu\left(t\right)\right]$ for $0 \le t \le \hat{t}$ [corresponding to the range $\hat{\mu} \le \mu\left(t\right) \le \mu_0$], and as the numeric solution of (76) with $\chi\left[\mu\left(t\right)\right] = \chi^*\left[\mu\left(t\right)\right]$ for $t \ge \hat{t}$ [corresponding to the range $\mu^* \le \mu\left(t\right) \le \hat{\mu}$]. Figure 4(a) plots the whole trajectory $\mu\left(t\right)$ for $0 \le t \le 400$, by distinguishing the first part (in grey), obtained through $\chi^s\left(\cdot\right)$ for $0 \le t \le \hat{t} = 36$, from the trajectory eventually converging to $\mu^* = 80.6452$ (in black) obtained by means of $\chi^*\left(\cdot\right)$ for $t \ge 36$.

The time-path trajectory $\chi\left(t\right)$ is then computed by evaluating the optimal policy (75) on the trajectory $\mu\left(t\right)$ just obtained, *i.e.*, by letting $\chi\left(t\right) = \chi\left[\mu\left(t\right)\right]$ for all $0 \le t \le 400$. Figure 4(b) reports the result once again emphasizing the first part (in grey) obtained through $\chi^s\left(\cdot\right)$ for $0 \le t \le \hat{t} = 36$, while that eventually converging to $\chi^* = 6.4516$ (in black) is obtained by means of $\chi^*\left(\cdot\right)$ for $t \ge 36$. In $t = 0$, the initial value is $\chi\left(0\right) = \chi_0 = 17.1194$, corresponding to $\mu_0 = 251.977$, while in $t = \hat{t} = 36$, $\chi\left(36\right) = 11.3688$; clearly, $\chi\left(\hat{t}\right) = 11.3688 < 12.5 = \hat{\chi}$, as expected.

7.2 Optimal policy of the original model

With the trajectories $\mu\left(t\right)$ and $\chi\left(t\right)$ at hand, we first compute the optimal consumption $\tilde{c}\left(A\right)$ and the optimal output $\tilde{y}\left(A\right)$ along the turnpike $\tilde{k}\left(A\right)$ in the original model, both as functions of the stock of knowledge A. By using (49) we immediately find the initial stock of knowledge $A_0 = 1.9707$ in $t = 0$, corresponding to $\mu_0 = 251.977$ established in the previous section by choosing $\hat{t} = 36$. To such value corresponds an initial endowment of capital $k_0 = \tilde{k}(A_0) = 496.57$ in $t = 0$; therefore, the starting point of the optimal trajectories on the turnpike is $(A_0, k_0) = (1.9707, 496.57)$.

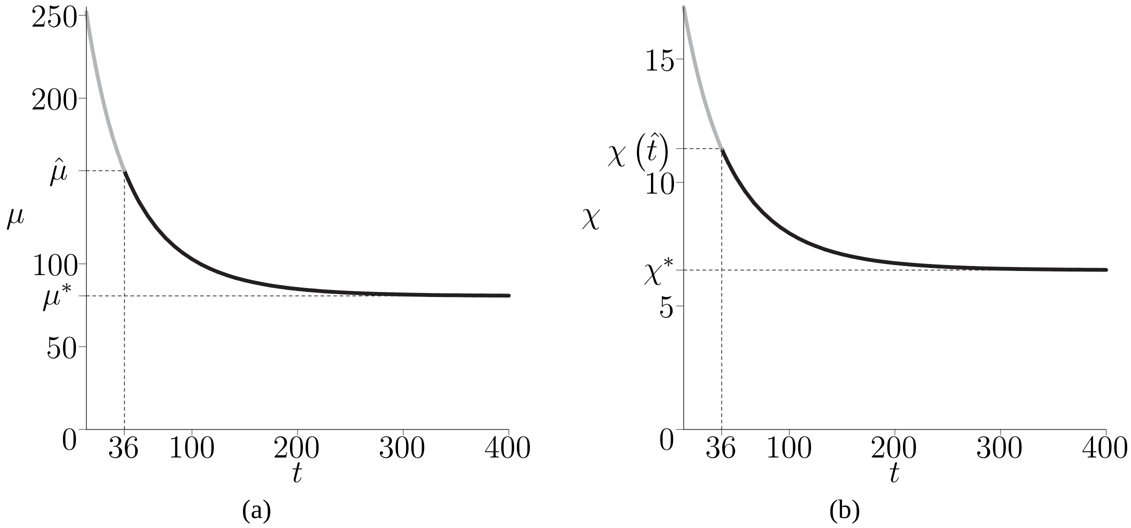

FIGURE 4: time-path trajectories of the detrended variables, $\mu(t)$ in (a) and $\chi(t)$ in (b), for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

The optimal consumption $\tilde{c}(A)$ along the turnpike is then obtained by using (50), in which $\chi(\cdot)$ is defined in (75), that is, $\chi^s(\cdot)$ for $A_0 \leq A \leq \hat{A}$ [corresponding, by (49), to $\hat{\mu} \leq \mu \leq \mu_0$], and $\chi^*(\cdot)$ for $A \geq \hat{A}$ [corresponding, by (49), to $\mu^* \leq \mu \leq \hat{\mu}$], where the abscissa of the intersection point between the turnpike $\tilde{k}(A)$ and the stagnation line $\hat{k}(A)$, corresponding to $\hat{\mu}$ in the detrended model, is, again by (49), $\hat{A} = 2.567$.

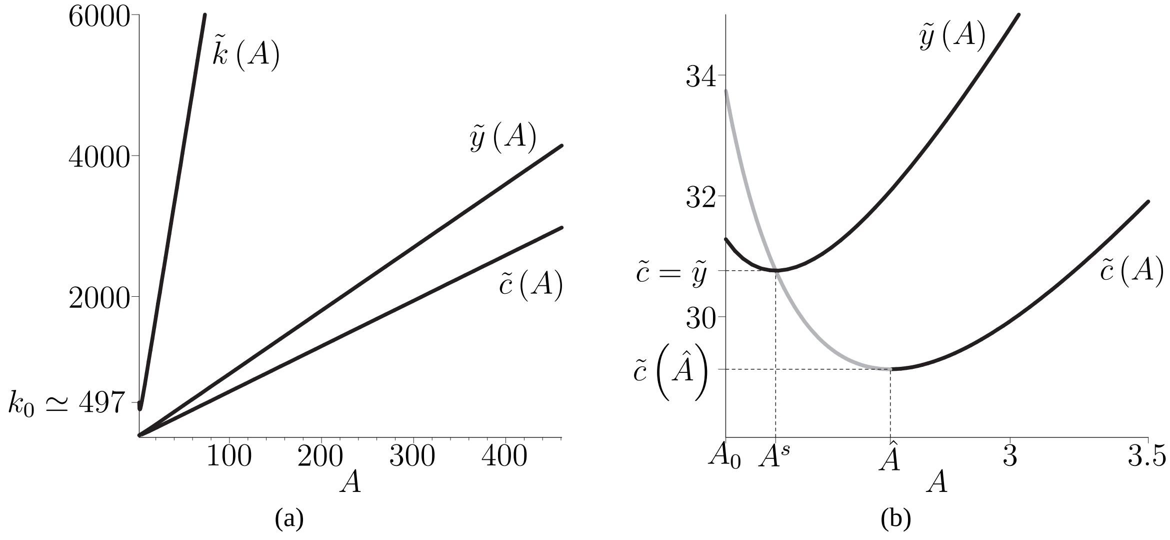

FIGURE 5: (a) optimal consumption, output and capital as functions of A along the turnpike $\tilde{k}(A)$ for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$; (b) optimal consumption and output close to the initial stock of knowledge $A_0 = 1.9707$.

Figure 5(a) reports the turnpike $\tilde{k}(A)$ as in Figure 2 plus the optimal output $\tilde{y}(A)$ corresponding to $\tilde{k}(A)$ and the optimal consumption $\tilde{c}(A)$ just evaluated. Figure 5(b) magnifies the intersection point between the optimal output $\tilde{y}(A)$ and the optimal consumption $\tilde{c}(A)$ occurring on $A^s = 2.1514$, close to the initial stock $A_0 = 1.9707$ and to the left of $\hat{A} = 2.567$, as discussed in Section 5.1. Since, through its counterpart μ^s in the detrended model, such intersection point has been used to construct the optimal policy for the initial values of the stock of knowledge A – precisely for $A_0 \leq A \leq \hat{A}$, corresponding to $\chi^s(\cdot)$ in the detrended model – in Figure 5(b) the graph of $\tilde{c}(A)$ between A_0 and $\hat{A}$ has been emphasized in grey, as we did in previous figures.

7.3 Time-path trajectories of the original variables

The time-path trajectory of the stock of knowledge, $A(t)$, is immediately obtained by evaluating (49) at each point of the time-path trajectory $\mu(t)$ computed in Section 7.1; hence, the time-path trajectories of capital, $\tilde{k}(t)$ and output, $\tilde{y}(t)$ along the turnpike follow by construction. As far as the optimal consumption along the turnpike, $\tilde{c}(t)$, is concerned, its time path-trajectory can be computed through (50) evaluated at each point of the trajectory $A(t)$ and with $\chi(\cdot)$ defined as in (75); specifically, by using $\chi^s(\cdot)$ for $0 \leq t \leq \hat{t} = 36$ and $\chi^*(\cdot)$ for $t \geq \hat{t} = 36$.

These trajectories are drawn in Figure 6(a), while in Figure 6(b) the time path-trajectory of the capital rental rate r is reported; once more, their dependence on the $\chi^s(\cdot)$ arm of the policy in (75) for $0 \leq t \leq \hat{t} = 36$ is emphasized in grey.

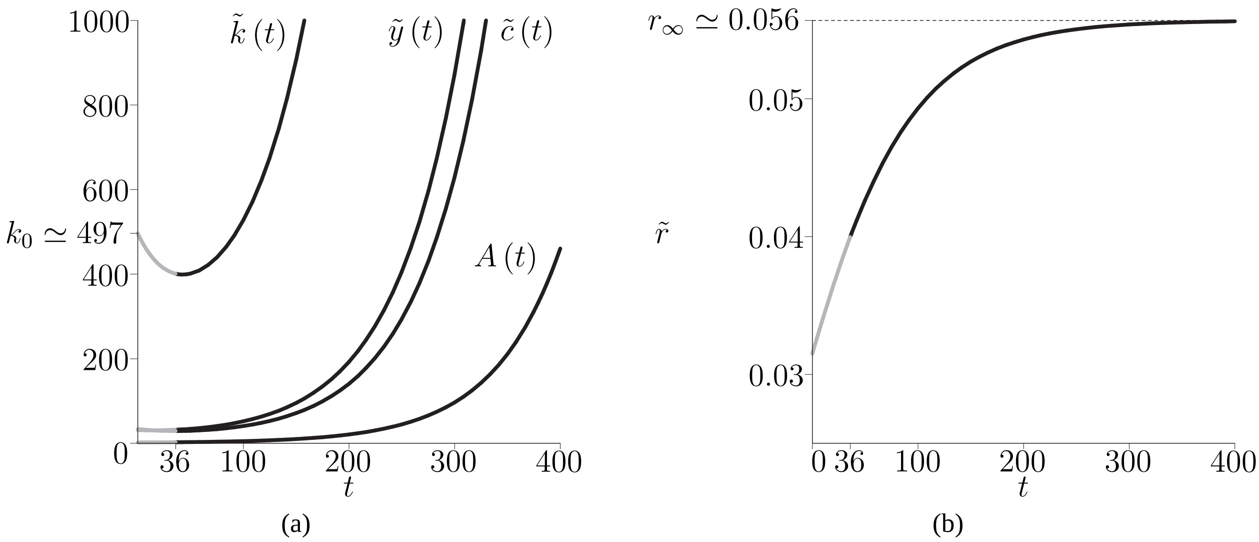

FIGURE 6: (a) time-path trajectories for the stock of knowledge A, capital $\tilde{k}$, output $\tilde{y}$ and optimal consumption $\tilde{c}$ along the turnpike for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$; (b) time-path trajectory for the capital rental rate $\tilde{r}$.

From Figures 2, 5(a) and 6(a), it is immediately seen that, under our choice for the parameters' values, the dynamics along the turnpike are characterized by a much larger amount of physical capital than any other variable. Specifically, a large initial capital $k_0 = 496.57$, compared to very few initial ideas available, $A_0 = 1.9707$, is required in order to let the Weitzman recombinant process to take-off; such amount of capital, even if for a short time, is partially being 'eaten up' by both consumption [recall that $\tilde{c}(A) > \tilde{y}(A)$ for $A_0 \leq A \leq A^s$] and investment in R&D, thus envisaging an initial period of decline for the physical capital $\tilde{k}$. As it is clear from Figure 5(b), also output and consumption decrease for a short time; specifically, output declines until $\tilde{c}(A)$ hits $\tilde{y}(A)$ from above – that is, when the stock of knowledge reaches level A^s – and consumption decreases until the stock of knowledge reaches level $\hat{A}$ (at the crossing point between the turnpike and the stagnation line, as seen from Figure

2) at time $\hat{t} = 36$. As time keeps elapsing, however, all variables start to increase, with the amount of capital $\tilde{k}$ keeping much higher values with respect of all others, especially the stock of knowledge A. For example, when A is around 73, $\tilde{k}$ is around 6000, as can be evinced from figure 5(a).

Especially Figure 5(a) emphasizes the striking high values for the ratio $\tilde{k}(A)/A$ – also ratios $\tilde{y}(A)/A$ and $\tilde{c}(A)/A$, however, are quite larger than 1 – which becomes constant as A increases, ***i.e.***, as the turnpike $\tilde{k}(A)$ approaches the asymptotic turnpike $\tilde{k}_{\infty}(A)$ in Figure 2 [note that all graphs, $\tilde{k}(A)$, $\tilde{y}(A)$ and $\tilde{c}(A)$ tend to become linear for large A].

In our example, thus, sustained long-run growth requires a large exploitation of physical resources, at least relatively to the other input factor, knowledge, even under a 'balanced' Cobb-Douglas technology assigning the same weight ($\alpha = 0.5$) to both its input factors (capital $\tilde{k}$ and knowledge A respectively). Such 'asymmetry' is explained by the ratio of between the (low) price of capital – which is 1, the numéraire – and the relatively high unit cost of knowledge production, $\varphi(A)$, as defined in (39); as a matter of fact, under our choice of $\beta = 0.0124$, $\varphi(A)$ turns out to be significantly larger than 1, as $\varphi(A) > \lim_{A\to\infty}\varphi(A) = 1/\pi'(0) = 1/\beta = 80.6452$ [see also Figures 8(a) and 8(b) in the next section].

We now focus our attention on the time-dynamics shown in Figure 6(a). The figure exhibits a system which actually takes some time to take-off, before starting to approach the constant balanced growth pattern in the long-run. In other words, provided that our economy starts with very few ideas available (less than two: $A_0 = 1.9707$) and with sufficiently large physical capital ($k_0 = 496.57$), the initial transient dynamics happen to last quite a bit; especially the stock of knowledge $A(t)$ in Figure 6(a) takes no less than 200 periods before becoming significant [note, however, that in the meantime capital $\tilde{k}(t)$ already started to "blow up"]; for example, it takes around 282 periods to reach the stock $A = 73$, corresponding to the amount of capital $\tilde{k} \simeq 6000$ discussed before. Similarly, the apparent constant ratio $\tilde{c}(A)/\tilde{y}(A)$ visible in Figure 5(a) – due to almost linearity of the functions $\tilde{c}(A)$ and $\tilde{y}(A)$, and which can be easily checked to be close to the asymptotic ratio 0.07184, corresponding to the asymptotic saving rate $s_{\infty} + s^{k}_{\infty} = 0.2816$ – is actually not reached before, say, at least 300 periods. To conclude, Figures 2 and 5(a) should be read carefully when one introduces time, as Figure 6(a) explains: of course the whole system grows along the turnpike $\tilde{k}(A)$, but at a very slow pace on the initial portion of it, while it keeps accelerating as time elapses until it "explodes" along the asymptotic turnpike $\tilde{k}_{\infty}(A)$.

Figure 6(b) adds another interesting piece of information to the analysis above: even if physical capital is always (much) larger than the stock of knowledge, its productivity keeps rising in time, as confirmed by its increasing rental rate, $\tilde{r}$, until it reaches its asymptotic value, $r_{\infty} = 0.0557$.

Finally, Figure 7 confirms all previous results in terms of rates of growth. The stock of knowledge $A(t)$ happens to be (by construction) the only variable with always positive rate of growth $\gamma_A = \dot{A}/A$; conversely, as already discussed before, the capital $\tilde{k}(t)$, the output $\tilde{y}(t)$ and consumption $\tilde{c}(t)$, all experience some negative growth at initial times, as they exhibit negative rates of growth, $\gamma_{\tilde{k}} = \dot{\tilde{k}}/\tilde{k}$, $\gamma_{\tilde{y}} = \dot{\tilde{y}}/\tilde{y}$ and $\gamma_{\tilde{c}} = \dot{\tilde{c}}/\tilde{c}$, for t close to zero. Interestingly, it can be observed that $\tilde{c}(t)$ reaches its absolute minimum in $\hat{t} = 36$ [corresponding to $\tilde{c}\left(\hat{A}\right)$, as confirmed by Figure 5(b)].

The most important feature of recombinant endogenous growth models, however, is strikingly evident in Figure 7: all rates of growth must be increasing in time, while approaching their asymptotic common rate $\gamma = 0.0157$, corresponding to balanced (and extremely fast in time) growth along the asymptotic turnpike $\tilde{k}_{\infty}(A)$. Such property is clearly consistent with the strictly convex shape of all curves in Figure 6(a). This feature reflects the original hypothesis introduced by Weitzman (1998): at initial periods, seed ideas are scarce, and thus have the potential of growing at increasing rates, while in the long-run, limited physical resources to be invested in R&D – with respect to the exploding number of seeds ideas available for matching – cools down growth to the more realistic case of

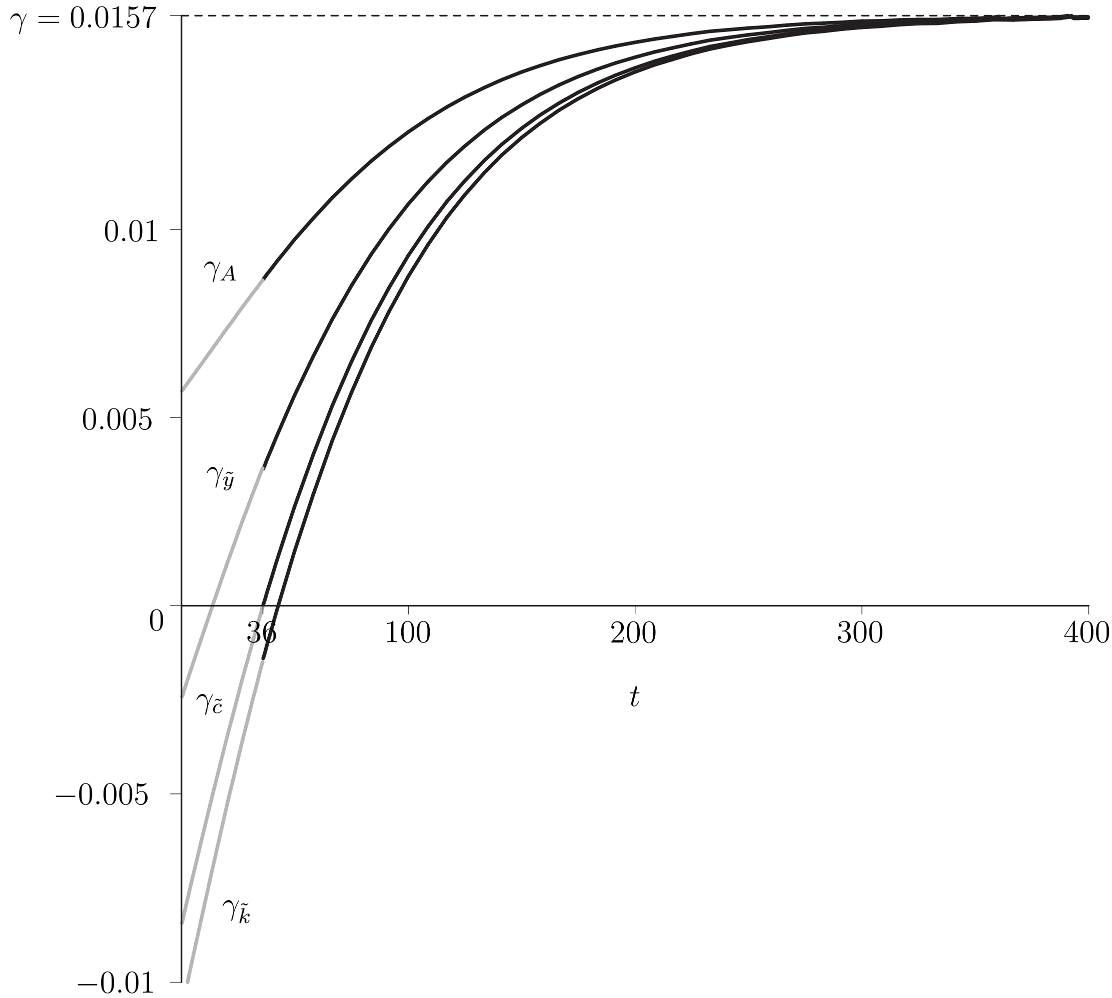

FIGURE 7: growth rates γ_A, $\gamma_{\tilde{k}}$, $\gamma_{\tilde{y}}$ and $\gamma_{\tilde{c}}$, of A, $\tilde{k}$, $\tilde{y}$ and $\tilde{c}$ respectively as functions of time, for $\alpha = 0.5$, $\rho = 0.04$, $\theta = \sigma = 1$ and $\beta = 0.0124$.

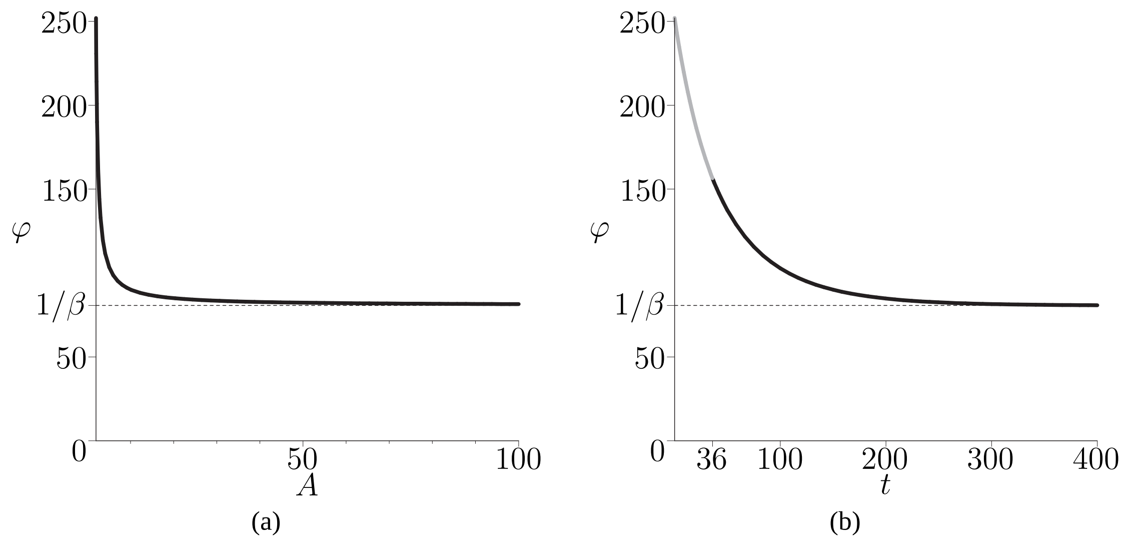

FIGURE 8: (a) expected unit cost of knowledge production, φ, as a function of the stock of knowledge A and (b) its time-path trajectory.

constant rates. In the next section we analyze more in detail the specific nature of the transition dynamics related to knowledge.

7.4 The dynamics of recombinant knowledge

Figure 8(a) shows the graph of the unit cost of knowledge production, $\varphi(A)$, given by (39) as a function of the stock of knowledge A: it is the arm of a hyperbole sharply decreasing for values of A close to A_0 with its asymptote given by $1/\pi'(0) = 1/\beta = 80.6452$. Such sudden jump, however, is to be diluted when time comes into the analysis; we have seen that the stock of knowledge A starts to grow significantly only after a certain amount of time [see Figure 6(a)], this fact explains why the hyperbole representing φ in Figure 8(b) as a function of time t – obtained by using the trajectory computed in the previous section, $A(t)$, in (39) – looks less steep than that in Figure 8(a). Note that in Figure 8(b), as well as in the following ones, we continue to emphasize in grey the portion of time-path trajectory dependent on the $\chi^s(\cdot)$ part of the policy in (75), that is, for $0 \leq t \leq \hat{t} = 36$.

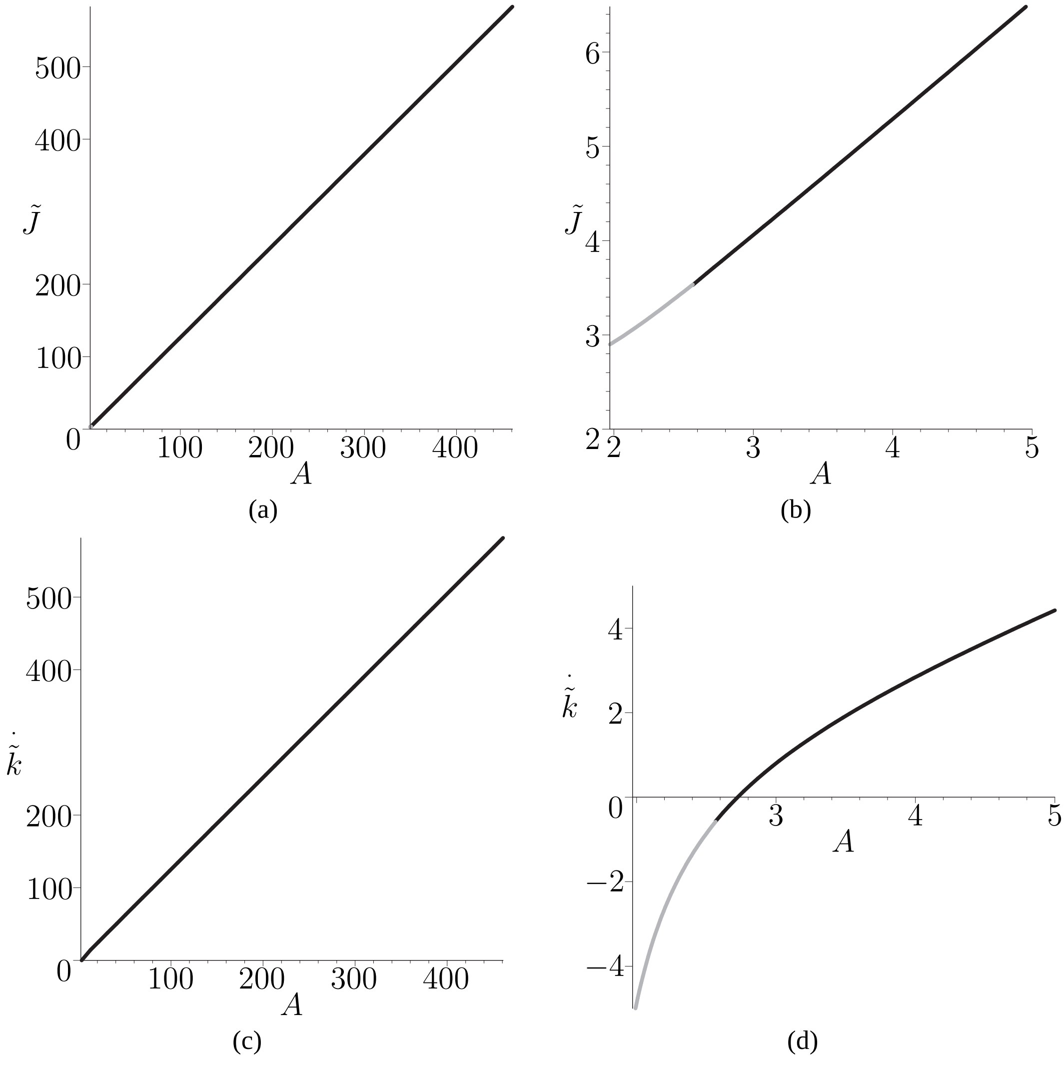

FIGURE 9: (a) investment in R&D, $\tilde{J}$, as a function of the stock of knowledge A and (b) its detail for A close to $A_0 = 1.9707$; (c) capital investment, $\dot{\tilde{k}}$, as a function of A and (d) its detail for A close to A_0.

Investment in R&D, $\tilde{J}$, and investment in physical capital, $\dot{\tilde{k}}$, along the turnpike as functions of the stock of knowledge, A, are plotted in Figure 9; $\tilde{J}$ is computed as in (22) by using functions $\tilde{c}(A)$ and $\tilde{y}(A)$ discussed in Section 7.2, and functions $\varphi(A)$ defined in (39) and $\tilde{k}'(A)$ obtained by differentiating (40) with respect to A. From Figures 9(a) and 9(c), where a large range of A values is considered, we learn that both $\tilde{J}$ and $\dot{\tilde{k}}$ are essentially linear functions of A; moreover, it is clear that both $\tilde{J}$ and $\dot{\tilde{k}}$ have the same magnitude, implying that they become the same well before reaching their asymptotic (common) constant rate, $s_\infty = J_\infty/y_\infty = s^k_\infty = \dot{k}_\infty/y_\infty = 0.1408$ (see Section 6). Only for A sufficiently close to its initial value, $A_0 = 1.9707$, their behavior differ, since $\dot{\tilde{k}}$ is negative for small A, when capital experiences initial 'disinvestment', as magnified by Figures 9(b) and 9(d).

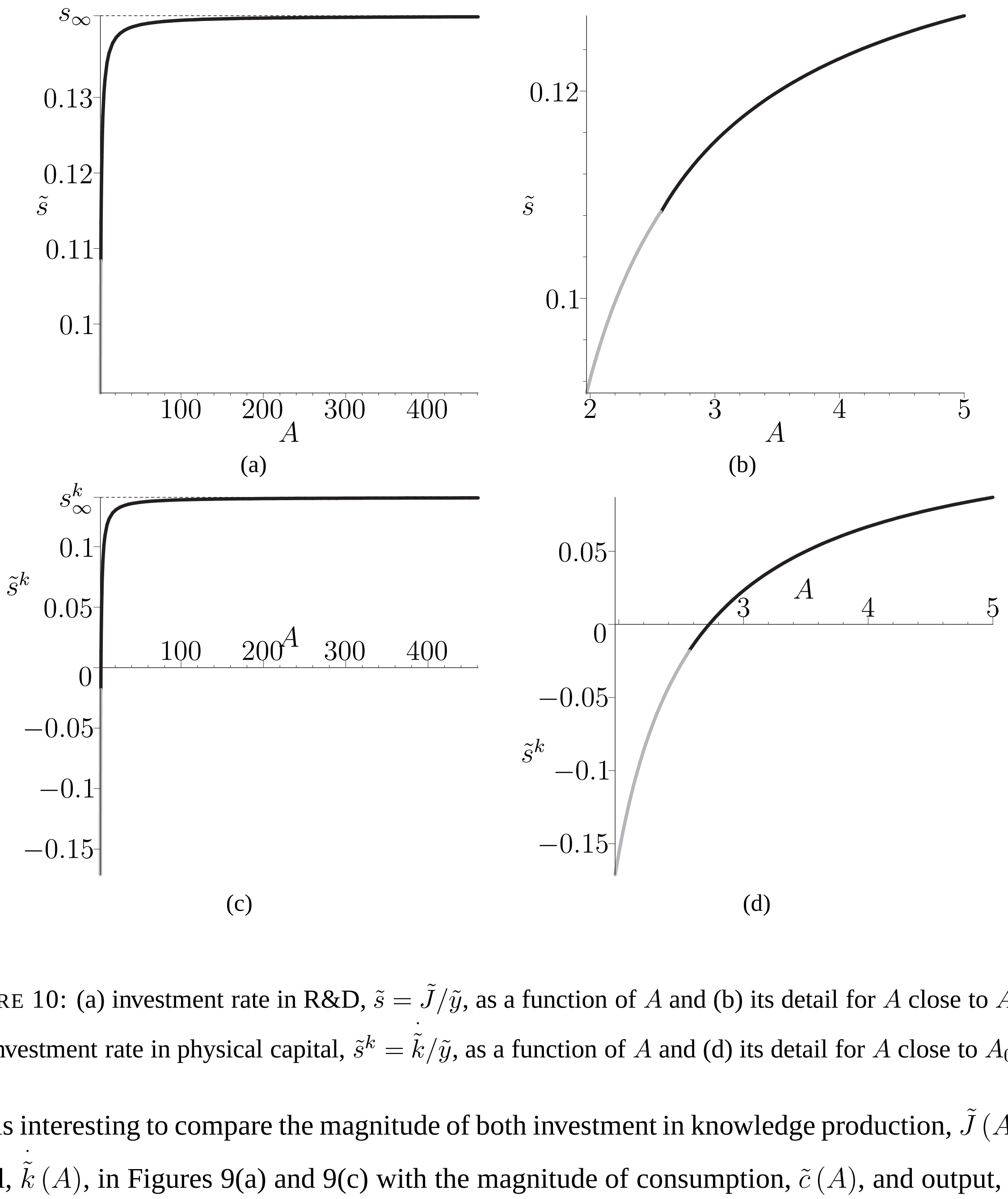

FIGURE 10: (a) investment rate in R&D, $\tilde{s} = \tilde{J}/\tilde{y}$, as a function of A and (b) its detail for A close to A_0; (c) investment rate in physical capital, $\tilde{s}^k = \dot{\tilde{k}}/\tilde{y}$, as a function of A and (d) its detail for A close to A_0.

It is interesting to compare the magnitude of both investment in knowledge production, $\tilde{J}(A)$, and capital, $\dot{\tilde{k}}(A)$, in Figures 9(a) and 9(c) with the magnitude of consumption, $\tilde{c}(A)$, and output, $\tilde{y}(A)$, in Figures 5(a) and 5(b): for all values of A – also close to A_0 – the optimal dynamics postulate a relatively small investment in both production factors with respect to consumption and output levels. To examine this property more in depth, Figures 10(a) and 10(c) report the investment rates in knowledge production and in physical capital as the ratios $\tilde{s} = \tilde{J}/\tilde{y}$ and $\tilde{s}^k = \dot{\tilde{k}}/\tilde{y}$ respectively, again as functions of the stock of knowledge A. Both investment rates are increasing in A and reach their asymptotic (common) value, $s_\infty = s_\infty^k = 0.1408$, quite rapidly, as confirmed by the details close to A_0 plotted in Figures 10(b) and 10(d), even if the investment rate in capital, $\tilde{s}^k$, is negative for small A, due to the initial disinvestment, as shown in Figures 10(c) and 10(d). Such quick jumps to the asymptotic value $s_\infty = s_\infty^k$ for both investment rates, $\tilde{s}$ and $\tilde{s}^k$, is consistent with the linearity exhibited by $\tilde{J}(A)$ and $\dot{\tilde{k}}(A)$ in Figures 9(a) and 9(c).

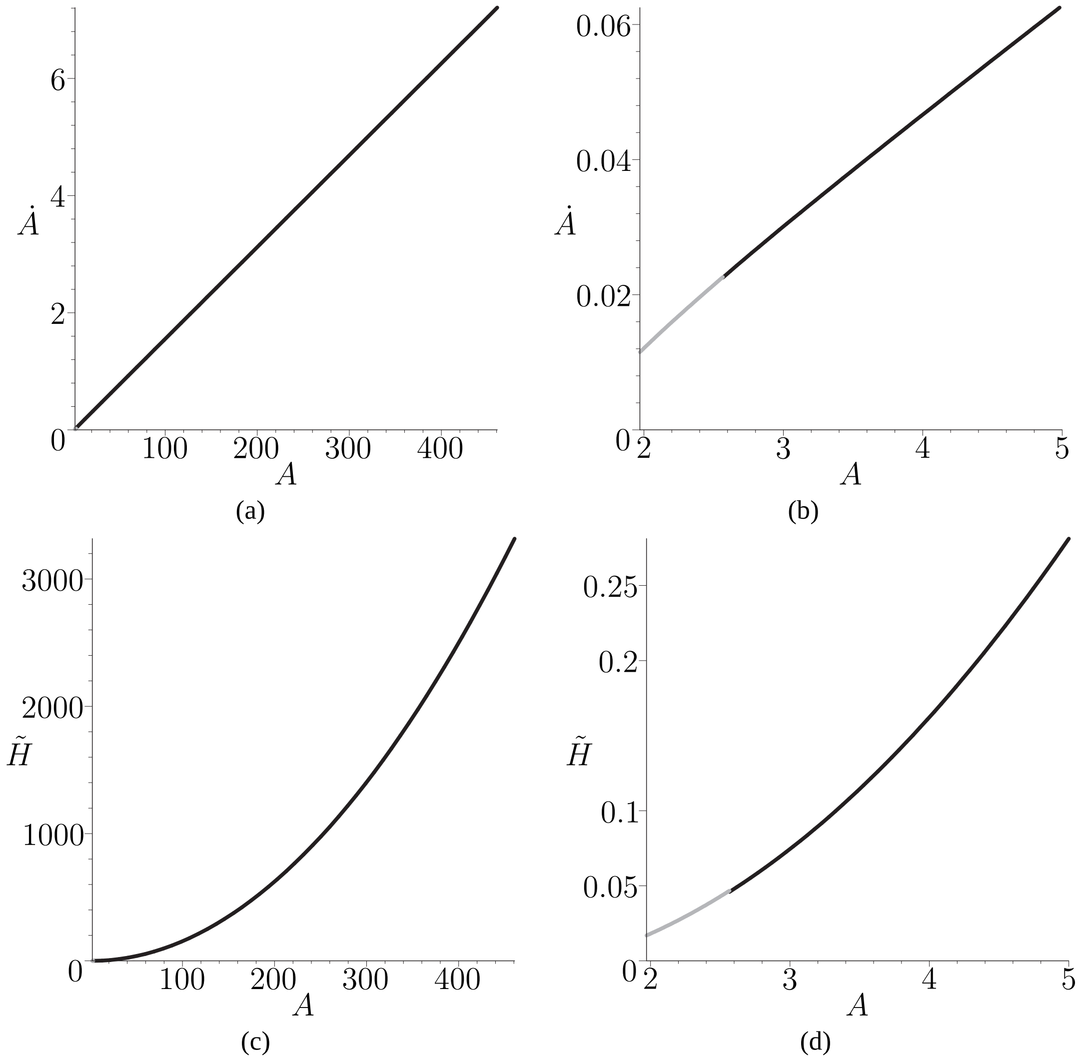

FIGURE 11: (a) new knowledge production, $\dot{A}$, as a function of A and (b) its detail for A close to A_0; (c) number of seed ideas, $\tilde{H}$, as a function of A and (d) its detail for A close to A_0.

As far as investment in knowledge, $\tilde{J}$ (or $\tilde{s}$), is concerned, these dynamics confirm Weitzman's (1998) description of the evolution of (recombinant) knowledge: when knowledge – and thus seed ideas H – is scarce the Weitzman's production function (5) exhibits low productivity; accordingly, only a small fraction of resources is employed in R&D, while such fraction increases as the stock of knowledge A – and thus seed ideas H – become more abundant. In the long-run, however, are the physical resources that become scarce with respect to knowledge – more specifically, they grow slower than what (potentially) could do knowledge – and it is this scarcity that bounds the (initially increasing) rate of investment $\tilde{s}$ to its asymptotic value s_∞, thus also bounding the whole economy to its long-run constant balanced growth-path.

With $\tilde{c}(A)$ and $\tilde{y}(A)$ at hand, it is also possible to evaluate the new (successful) knowledge production, $\dot{A}$, and the evolution of seed ideas, $\tilde{H}$, along the turnpike as functions of the stock of knowledge, A; the former is given by (29), while the latter can be computed directly from the Weitzman's dynamics (4), where $\dot{A}$ has just been obtained and $C_2'(A) = (\partial/\partial A)[A(A-1)/2] = A - 1/2$. Their graphs are reported in Figure 11, in which it is striking the linearity of $\dot{A}$, even for values of A close to its initial value, A_0, as can be grasped from Figure 11(a) and, especially, the detail in Figure 11(b), while the seed ideas $\tilde{H}$ remain uniformly convex for all values of A, also when A is close to A_0, as Figures 11(c) and 11(d) clearly show. Strict convexity of $\tilde{H}(A)$ in Figures 11(c) and 11(d), associated to (more or less uniform) linearity of $\dot{A}$ in Figures 11(a) and 11(b), is consistent to formula (4) – with $C_2'(A) = A - 1/2$ itself linear – which implies quadratic growth for $\tilde{H}$ when $\dot{A}$ grows linearly. It is also worth noting the difference in magnitudes between the number of seed ideas $\tilde{H}$ produced for each given stock A and the actual successful ideas $\dot{A}$ produced out of $\tilde{H}$, even for small values of A: such low returns are justified by the choice of a very low value for the efficiency parameter, $\beta = 0.0124$, in the probability of success (38) of the Weitzman's matching process; with such a low β, seed ideas $\tilde{H}$ must abound in order to guarantee sustained growth of knowledge.

To conclude, Figure 12 shows the time-path trajectories of all variables just discussed, specifically, $\tilde{J}$, $\dot{\tilde{k}}$, $\tilde{s}$, $\tilde{s}^k$, $\dot{A}$ and $\tilde{H}$. Due to the slow growth pace of the stock of knowledge $A(t)$ for initial periods, as reported in Figure 6(a), linearity of investments $\tilde{J}(A)$, $\dot{\tilde{k}}(A)$ and new knowledge $\dot{A}$ as functions of A, evident in Figures 9(a), 9(c) and 11(a) respectively, gives way to corresponding time-path trajectories which are convex, as shown in Figures 12(a), 12(b) and 12(e), while, for the same reason, convexity of the seed ideas $\tilde{H}(A)$ as a function of A in Figure 11(c) becomes more accentuated in its correspondent time-path trajectory of Figure 12(f). Similarly, the sudden jumps to their asymptotic values of the investment rates $\tilde{s}(A)$ and $\tilde{s}^k(A)$ as functions of A exhibited in Figures 10(a) and 10(c) respectively, is being smoothed down in their corresponding time-path trajectories of Figures 12(c) and 12(d), again by the initial slow growth of the stock of knowledge $A(t)$; that is, along their time-path trajectories both investment rates need at least 200 periods before they start approaching their long-run (common) constant value $s_\infty = s_\infty^k = 0.1408$.

8 Conclusions

The exercise performed in this paper is a very preliminary attempt to tackle the transition dynamics in the recombinant growth model introduced by Tsur and Zemel (2007). For CIES instantaneous utility and Cobb-Douglas production in the output sector, we chose a suitable function for the Weitzman's (1998) probability of obtaining a successful idea from each pairwise matching of seed ideas, so that the original optimal dynamics along the turnpike, which diverge at a constant rate of growth in the long-run, can be 'detrended' to an equivalent system converging to a steady state. In the space of the detrended variables we exploit the asymptotic steady state plus a singular point, across which the optimal policy must get through at some early instant, in order to compute numerically two optimal trajectories which, for a specific choice for the parameters' values, happen to match on a large range between such two points. We therefore conclude that, by joining together these trajectories, we can build an approximation of the optimal policy in the detrended variables which must be reasonably close to the true policy for all feasible values of the detrended variables. By converting such trajectory into the original state variable (stock of knowledge) and control variable (consumption) trajectory along the turnpike, we obtain a numeric approximation of the optimal consumption, which in turn,

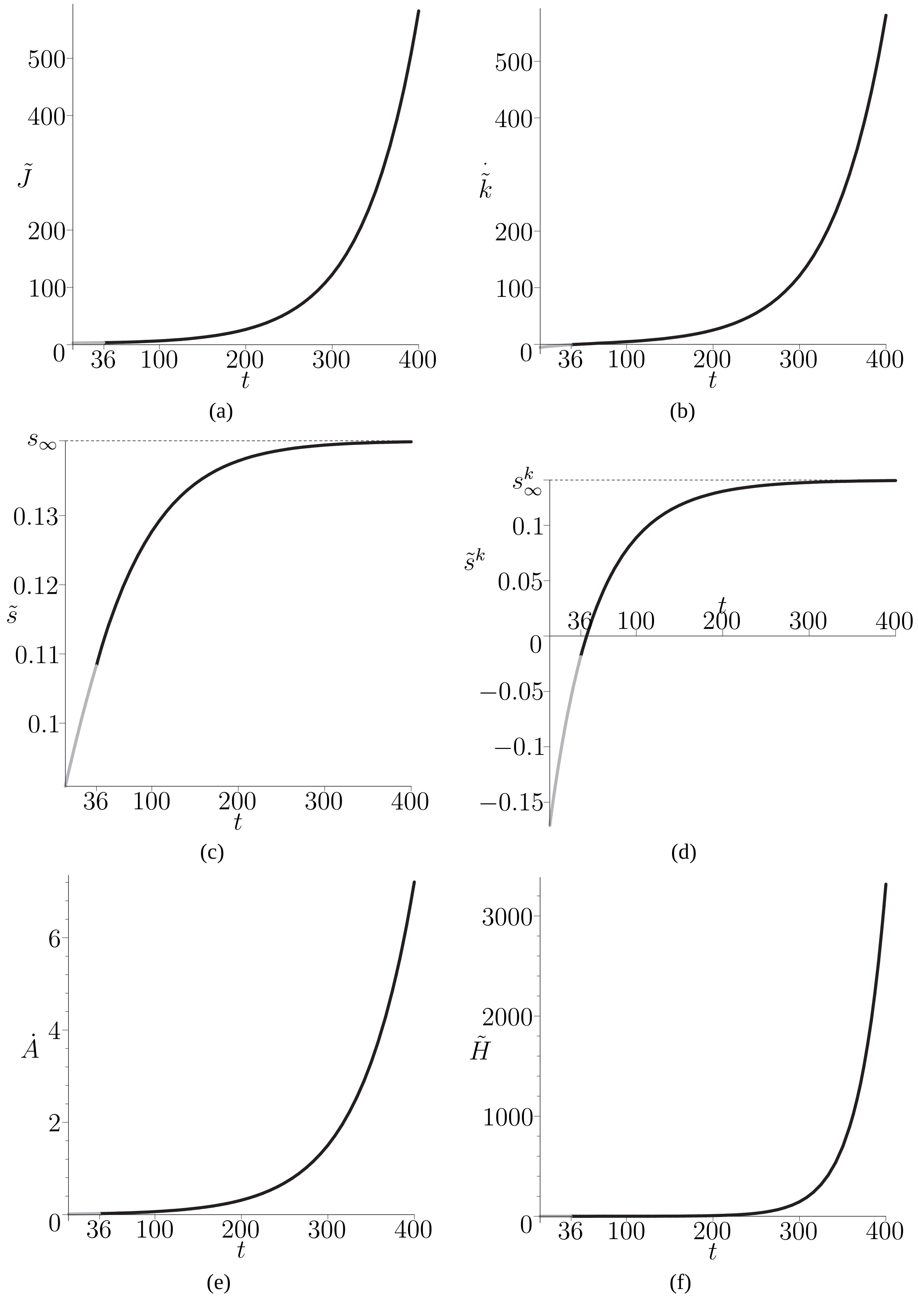

FIGURE 12: (a) time-path trajectory of the investment in R&D, $\tilde{J}$, (b) time-path trajectory of the capital investment, $\dot{\tilde{k}}$, (c) time-path trajectory of the investment rate in R&D, $\tilde{s} = \tilde{J}/\tilde{y}$, (d) time-path trajectory of the investment rate in physical capital, $\tilde{s}^k = \dot{\tilde{k}}/\tilde{y}$, (e) time-path trajectory of the new knowledge production, $\dot{A}$, (f) time-path trajectory of the number of seed ideas, $\tilde{H}$.

again by solving numerically an ODE, yields the transition optimal time-path trajectories of the stock of knowledge, physical capital, output and consumption – as well as their transition growth rates – along the turnpike.

We believe that the main technical contribution of the present work is the appropriate form chosen for the Weitzman's probability function defined in Assumption A.4(ii), which allows for 'detrending' the original system of ODEs (37) into the equivalent system (58).

If, on one hand the optimal policy obtained in section 6 – and used to build time-path trajectories in Section 7 – may clearly be of interest per se, on the other hand it is insufficient for studying how the system behavior along the transitional turnpike is being affected by changes in the technological parameter β of the probability function π of Assumption A.4(ii), while keeping fixed the values of the other parameters. In order to further pursue the analysis toward this direction, one needs either to improve the numerical computation of system (58) so that the matching of the two aforementioned trajectories in the detrended space – one crossing at the asymptotic steady state and the other crossing the singular point – is maintained at least on a nontrivial interval of values for parameter β for given values of the other parameters, or trying a completely different approach on either system (37) or system (58) by means of analytical tools in order to explicitly find the true form of the optimal trajectories. One may tackle the latter by looking for some special function that may prove useful in solving one of the systems (37) or system (58); see, *e.g.*, Boucekkine and Ruiz-Tamarit (2008) for a recent application of the Gaussian hypergeometric functions to the Lucas-Uzawa model. Both approaches will be investigated in future research projects.

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