On Fragility of Bubbles in Equilibrium Asset Pricing Models of Lucas-Type[1]

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Abstract

In this paper we study the existence of bubbles for pricing equilibria in a pure Exchange Economy à la Lucas, with infinitely lived homogeneous agents. The model is analyzed under fairly general assumptions: no restrictions either on the stochastic process governing dividends' distribution or on the utilities (possibly unbounded) are required. We prove that the pricing equilibrium is unique as long as the agents exhibit uniformly bounded relative risk aversion. A generic result of uniqueness is also given regardless of agent's preferences. Several "pathological" examples exhibiting equilibrium prices with bubble components are constructed. Finally, the presence of ambiguous bubbles along the theory developed by Santos and Woodford is studied by means of a transversality condition at infinity. The whole discussion sheds more insight on the common belief that bubbles are a marginal phenomenon in such models.

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1 Introduction

The main objective of this paper is to test how reasonable is the conjecture that multiple equilibria, or bubbles[2], are a negligible phenomenon in sequential equilibrium models with infinitely lived homogeneous agents of Lucas-type [12]. Surprisingly, this issue has not attracted much attention, at least in a general formulation, where no restrictions both on preferences and on the probabilistic structure governing the process of future dividends are imposed. On the other hand, relatively few examples of bubbles can be found in the literature. At least two reasons may explain this "lack of interest". First, since all bubble-producing factors are absent in Lucas model, it has been taken for granted that they should emerge only in rather special circumstances. This view-point is easily captured by consulting the by now wide literature on intertemporal asset pricing models (see [17], [12], [3], [21], [4], [7], [11], [6], [13], [16], [8])[3]. Perhaps, this intuition suggested that economies with a representative agent in which the equilibrium allocation is given by the initial resources needed no further investigation, thus addressing more attention toward analyzing equilibrium models with heterogeneous agents as well as with various debt constraints.

A second reason for the scarce interest in studying price bubbles in Lucas' models is perhaps due to the well known analytical difficulties closely related to the formulation of some necessary condition of transversality at infinity (see Ekeland and Scheinkman [5]). Results available in literature show that in the stochastic setting this is a hard task unless severe restrictions are imposed (see [22] and [20]).

However, there exists a line of research that is worth mentioning: Epstein and Wang [6] generalize the original Lucas' model by including the representation of beliefs to take into account uncertainty aversion of the representative agent. They find equilibria that are indeterminate, and it is implicit in their analysis that such an approach is the sole way to get robust multiple price equilibria results.

Two recent papers stimulated the present research. Santos and Woodford [16] established general results on rational bubbles within a quite broad scenario of sequential equilibrium models where traders have rational expectations. They proved that perpetual assets in non-zero net supply cannot give rise to unambiguous price bubbles and, in addition, any sort of bubbles is excluded whenever the preferences satisfy a certain property of discounting[4]. However, it is important to stress that Santos and Woodford's analysis rests on the simplified assumption that the underlying stochastic environment has a tree structure with finitely many information sets at each instant of time. This allows them to provide an elegant theory of asset pricing which extends to a dynamic context Kreps' arbitrage approach. Besides this, it is very important to remark that their analysis is somewhat different than ours: rather than focussing on multiple equilibria, they are concerned with the issue of whether a given equilibrium pricing involves a bubble component. Indeed, in such a general model, a natural consequence of dynamical incompleteness of markets is that the present value of the streams of future dividends is not uniquely determined, causing several complications and additional types of bubbles (the so-called ambiguous bubbles). We discuss some of these aspects

[2] We are aware that our terminology may give rise to misinterpretations. We are mainly concerned with existence of multiple equilibria. Following the classical stand-point, the assets fundamental value is unambiguously defined, and thus bubble existence is equivalent to multiple pricing equilibria. However, after [16], the distinction between the fundamental value and the bubble component has become questionable. This aspect will be discussed later on.

[3] The work of Gilles and Leroy (see [7]) can be considered an exception. They actually stressed the possibility of coexistence of positive price bubbles in a model where a representative agent holds an asset forever and who does not resort on Ponzi schemes. However, their setting is different: they are concerned with Arrow-Debreu economies with complete forward markets and not with sequential equilibria we focus on.

[4] A similar result has been formulated by Magill and Quinzii [13].

that are in common with our results in Section 6.

The second paper is due to Kamihigashi [9]. To further strengthen the idea of marginality of bubbles, he provides a condition that assures the uniqueness of equilibrium in Lucas' model which is not properly related to discounting properties of agent's preferences. To construct an example of multiple equilibria in a two-period economy where there are countably many states of the world, he needs to assume that the consumer's utility function be unbounded. Moreover, he makes an important remark by observing that the presence of positive bubbles in his example is closely related to the violation of the Euler equation.

Originally, our motivation for investigating multiple equilibria in full information, pure exchange economies was born from the common belief that optimizing behavior of a representative agent with smooth preferences should be enough to rule out bubbles, regardless of the random behavior of the exogenous shocks. While we have not been lucky in establishing such a short-cut general result, in the present work we definitely provide robust arguments that confine the appearance of multiple-priced equilibria to a class of very special phenomena.

Most of the analysis carried out in our paper is in tune with the traditional approach that singles out non-existence of bubbles and existence of a unique equilibrium as equivalent concepts. Following [9], we implement Lucas' model on two important aspects. First, no restriction on the probabilistic law of dividends is postulated (a similar setting can be found in [10] as well, but for different purposes). Second, nearly no boundedness assumption on both dividends and prices is assumed, as well as on utilities. The only hypotheses maintained are the differentiability of preferences and the zero short-sales constraint. This severe restriction is made to eliminate as much as possible infinite arbitrage opportunity like Ponzi schemes. Therefore we deviate from the line of research, initiated by Kocherlakota [11], that emphasizes the form of the borrowing constraints as a bubble-producing factor (see also [16], [8]).

Owing to such a general setting, we need first to study carefully the consistency of the model, in order to formulate suitable necessary and sufficient conditions for price equilibria to exist such that they include fundamental values of assets. This is pursued in Section 3 where we show that the standard Euler equation for the model described in Section 2 is not necessary to construct the theory of equilibrium valuation for the assets. Indeed, in place of the stochastic Euler equation, we shall utilize an Euler inequality as a necessary condition for optima. In accordance with the observation in [9], it is soon realized that the imposition of the Euler equations is not fully justified and may preclude potential bubbles.

Section 4 is devoted to make precise the notion of "fragility" for potential multiple equilibria. After presenting Kamihigashi's result [9], which is a sufficient condition for uniqueness of the equilibrium, we proceed ahead by establishing a new result that characterize potential multiple pricing equilibria precisely as a borderline phenomenon: a slight modification of the amounts of assets, or that of dividends, has the effect that multiple equilibria disappear. To strengthen our argument, we then show that all preferences exhibiting uniformly bounded relative risk aversion fall outside the class of models with bubbles. Therefore, unlike Kamihigashi, we rely on agent's risk aversion behavior rather than bounded/unbounded property of utilities to specify uniqueness of equilibrium.

It is natural, after having outlined multiplicity of equilibria as a possible outcome only in a non-generic set of economies, to devote our attention to the study of this "tiny" category. We are able to provide a rough classification of bubbles into two categories: bursting and non-bursting bubbles. Both types violate Kamihigashi sufficient conditions of uniqueness and it turns out that unbounded relative risk-aversion preferences are the key ingredient in their construction. In Section 5 we present examples of the first kind that cover, in some sense, all those known in literature, while

Subsection 6.3 contains (more elaborated) examples of the second type which are novel.

Finally, most of Section 6 is dedicated to the study of a transversality condition at infinity that turns out to be closely related to the exclusion of Ponzi schemes. In this section, we eventually take into account the issue of ambiguous bubbles introduced by Santos and Woodford [16]. For instance, within our setting we are able to formulate an improved version of the discounting assumption which rules out potential bubbles with respect to any state prices consistent with the equilibrium.

Section 7 concludes by comparing our results with those of other authors and by specifying possible further developments of the present research. Most of the lengthy proofs are gathered in Section 8.

2 The Model

There are k productive assets, each in fixed supply, that produce random quantities of a single non-storable consumption good in all time periods. Consumers are identical in terms of utilities and endowments. At each trading time there are spot markets both for the consumption good and for shares in the assets. The uncertainty is modelled by a probability space $(\Omega, \mathbf{F}, \mu)$ where $\mathbf{F} = \{\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}\}$ is a filtration of σ-algebras describing the revelation of information. The asset dividends $\mathbf{d} = \left\{d_t(\omega) \in \mathbf{R}^k_+, t = 0, 1, 2, ...\right\}$ form an $\mathbf{F}$-adapted process which represents the amount of the consumption good yielded by one unit of each asset. The process $\mathbf{w} = \{w_t(\omega)\}$ is the non-negative $\mathbf{F}$-adapted process of exogenous endowments of the consumption good.

We shall write $\mathbf{E}_t(\cdot)$ instead of $\mathbf{E}(\cdot \mid \mathcal{F}_t)$. In general, the initial σ-algebra $\mathcal{F}_0$ may not be the trivial one and thus the operator $\mathbf{E}_0$ does not agree necessarily with the expectation $\mathbf{E}$[5].

Households' preferences are given by the separable life-time utility

$$\mathbf{E}_0 \sum_{t=0}^{\infty} u_t \left[c_t(\omega), \omega\right]$$

defined over the consumption processes $\mathbf{c} = \{c_t(\omega)\}$. Each instantaneous utility u_t is not necessarily uniform across states and the series needs not to be convergent. The process of assets holding strategy is denoted by $\mathbf{y} = \{y_t(\omega)\}$. The initial endowment of each asset is normalized to one, i.e., $y_0 = \mathrm{e} = (1, 1, ..., 1) \in \mathbf{R}^k$.

Here are the main assumptions to be effective throughout this paper. Even where it is not explicitly specified, properties pertaining all the functions involved must hold almost surely with respect to the measure μ. For vectors notation, a superscript will denote its component. For instance, $d^i_t(\omega)$ is the dividend payed by asset i, at epoch t when the state of the world is ω.

A. 1 $0 < d_t(\omega) \cdot \mathrm{e} + w_t(\omega) = \sum_{i=1}^k d^i_t(\omega) + w_t(\omega) < +\infty$ *a.s. for all* t.

A. 2 *For each* t*, utilities* $u_t(\cdot, \cdot)$ *are* $\mathcal{B}^1 \otimes \mathcal{F}_t$*-measurable, where* $\mathcal{B}^1$ *is the Borel* σ*-algebra in* $\mathbf{R}_+$*, and, for each fixed* ω*,* $u_t(\cdot, \omega)$ *are concave, strictly increasing and differentiable over* $\mathbf{R}_{++}$.

Assumption A.1 could be relaxed by admitting that the total good supply $d_t \cdot \mathrm{e} + w_t$ may vanish with positive probability. However this requires the marginal utility to be finite at zero, thus generating some further formal complications. Needless to say, Assumption A.2 encompasses

[5] This is not a merely empty generalization. It enables us to treat time t homogeneously. Every result obtained for $t = 0$ is immediately translated to any epoch t.

standard unbounded utilities, like logarithm, having $u_t(0,\omega) = -\infty$ with positive probability, as well as functions having infinite derivative at zero.

A contingent plan $(\mathbf{c}, \mathbf{y}) = \{c_t(\omega), y_t(\omega)\}$, $t \geq 0$, is said to be feasible if:

i) $c_t(\omega) \geq 0$ are $\mathcal{F}_t$-measurable variables for all $t \geq 0$;

ii) $y_t(\omega) \geq 0$ are $\mathcal{F}_{t-1}$-measurable for $t \geq 1$ and $y_0 = \mathrm{e} = (1, 1, ..., 1)$;

iii) $c_t(\omega) + p_t(\omega) \cdot [y_{t+1}(\omega) - y_t(\omega)] \leq d_t(\omega) \cdot y_t(\omega) + w_t(\omega)$ *a.s.* for $t \geq 0$.

Below we give the definition of Arrow-Radner sequential equilibrium, where Brock's [2] concept of weak maximality is adopted. To ease notation, from now on we will drop the argument ω of all the random functions under study. By abusing a bit notation, we shall also write $u_t(c_t)$ instead of $u_t(c_t(\omega), \omega)$ and the derivative $D_1 u_t(c,\omega)$ will be denoted by $u_t^{'}(c)$. Symbols X^- and X^+ will denote the negative and the positive part of random variable X, respectively. We also recall that, for non-negative random vectors $Y(\omega) \in \mathbf{R}^k$, the notation $\mathbf{E}_t(Y) < +\infty$ means $\mathbf{E}_t(|Y|) < +\infty$ or, equivalently, $\mathbf{E}_t(Y \cdot \mathrm{e}) < +\infty$. For a measurable set $A \in \mathcal{F}$, the indicator function of A will be denoted by $\mathbf{1}_A$.

Definition 1 *An equilibrium is an* $\mathbf{F}$*-adapted price process* $\mathbf{p}$ *such that:*

i) $0 \leq p_t < +\infty$ *a.s. for all t;*

and the plan $\mathbf{c}^* = \{c_t^*\} = \{d_t \cdot \mathrm{e} + w_t\}$, $\mathbf{y}^* = \{\mathrm{e}\}$ *satisfies the two conditions:*

ii) $\mathbf{E}_0 [u_t(c_t^*) - u_t(c_t)]^- < +\infty$ *a.s. for all t,*

iii) $\limsup_{N\to+\infty} \mathbf{E}_0 \sum_{t=0}^{N-1} [u_t(c_t^*) - u_t(c_t)] \geq 0$ *a.s.*

for any feasible plan $(\mathbf{c}, \mathbf{y})$ *where the* y_t*'s are essentially bounded, for each epoch* $t \geq 1$.

Here we are given the standard condition of no-trade equilibrium in which agents hold their assets forever, and consume all their available wealth $d_t \cdot \mathrm{e} + w_t$ at each trading date. The same framework adopted here has been used in [9]. We want to remark that the restriction concerning boundedness of the y_t's is needed for technical reasons. It will only be effective whenever sufficient conditions of optimality are used (see Proposition 2). It will also be seen that, at least for the fundamental prices, the plan $\mathbf{c}^*$ satisfies the stronger property of optimality $\liminf_{N\to+\infty} \mathbf{E}_0 \sum_{t=0}^{N-1} [u_t(c_t^*) - u_t(c_t)] \geq 0$. Henceforth, we shall always write $\{c_t^*\}$ to denote the equilibrium consumption path $\mathbf{c}^* = \{d_t \cdot \mathrm{e} + w_t\}$.

3 Existence of equilibria

The goal of this section is that of building the equilibrium analysis of our model on a solid basis. The starting point is formula (1) below, which turns out to be a first-order condition, that takes the form of an Euler inequality rather than equality. As a matter of fact, and as its proof will make clear, under our assumptions only the right-hand directional derivative is well defined.

Proposition 1 *Under A.1-2, if* $\mathbf{p}$ *is an equilibrium then:*

$$u_{t-1}^{'}\left(c_{t-1}^{*}\right)p_{t-1}\geq \mathbf{E}_{t-1}\left[u_{t}^{'}\left(c_{t}^{*}\right)\left(p_{t}+d_{t}\right)\right] \tag{1}$$

for $t\geq 1$.

One could ask in what cases equality in (1) is necessarily true. This requires to perform the left-hand derivative in the proof of this proposition. That is problematic insofar one must impose *a priori* restrictions on prices p_t. It is not difficult to show that (1) holds with equality if there are satisfied the following two conditions:

$$p_{t}\cdot \mathrm{e}\leq Mc_{t}^{*} \tag{2}$$

for some scalar M, and

$$\mathbf{E}_{t-1}\left[u_{t}\left(c_{t}^{*}\right)-u_{t}\left(\zeta d_{t}\cdot \mathrm{e}+w_{t}\right)\right]<+\infty \tag{3}$$

for some $\zeta<1$. For instance, either conditions are true when the states of the world are finite at trading date t. We shall also see later that a sufficient conditions for (3) is that u_t exhibits a bounded relative risk-aversion.

Inequality (1) will be enough to build up pricing analysis within our general setting. By iterating (1) starting from t, we get

$$u_{t}^{'}\left(c_{t}^{*}\right)p_{t}\geq \mathbf{E}_{t}\sum_{s=1}^{N}u_{t+s}^{'}\left(c_{t+s}^{*}\right)d_{t+s}+\mathbf{E}_{t}\left[u_{t+N}^{'}\left(c_{t+N}^{*}\right)p_{t+N}\right]$$

which, by taking the limsup over N, yields

$$u_{t}^{'}\left(c_{t}^{*}\right)p_{t}\geq \mathbf{E}_{t}\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^{*}\right)d_{t+s}+\limsup_{N\rightarrow+\infty}\mathbf{E}_{t}\left[u_{t+N}^{'}\left(c_{t+N}^{*}\right)p_{t+N}\right]. \tag{4}$$

Since, by (i) of Definition 1, $u_{t}^{'}\left(c_{t}^{*}\right)p_{t}<+\infty$ and the last term is non-negative, we infer both conditions (5) and (6) displayed in the following proposition.

Proposition 2 *Under A.1-2, a necessary condition for equilibria to exist is*

$$\mathbf{E}_{0}\sum_{t=1}^{\infty}u_{t}^{'}\left(c_{t}^{*}\right)d_{t}<+\infty. \tag{5}$$

In this case

$$u_{t}^{'}\left(c_{t}^{*}\right)p_{t}\geq \mathbf{E}_{t}\sum_{s=1}^{\infty}u_{t+s}^{'}\left(c_{t+s}^{*}\right)d_{t+s} \tag{6}$$

for all $t\geq 0$.

In view of (6), let us define

$$u_t^{'}(c_t^*)\, p_t = \mathbf{E}_t \sum_{s=1}^{\infty} u_{t+s}^{'}\left(c_{t+s}^*\right) d_{t+s} + u_t^{'}(c_t^*)\, b_t \tag{7}$$

where the "bubble" component $b_t(\omega) \in \mathbf{R}_+^k$ is $\mathcal{F}_t$-measurable. Consequently, we can define the market fundamental (adapted) process $\mathbf{f} = \{f_t\}$ as

$$f_t = \frac{1}{u_t^{'}(c_t^*)} \mathbf{E}_t \sum_{s=1}^{\infty} u_{t+s}^{'}\left(c_{t+s}^*\right) d_{t+s}. \tag{8}$$

Thanks to (7), if $\mathbf{p}$ is an equilibrium, then $\mathbf{p} = \mathbf{f} + \mathbf{b}$, with $\mathbf{b} = \{b_t\}$. We have here the traditional definition of speculative bubble as the difference between the price of the asset and its fundamental value. Clearly, the fundamental price process $\mathbf{f}$ satisfies Euler inequality (1) with equality, i.e.:

$$u_{t-1}^{'}\left(c_{t-1}^*\right) f_{t-1} = \mathbf{E}_{t-1}\left[u_t^{'}(c_t^*)\,(f_t + d_t)\right]$$

while the non-negative price bubble $\mathbf{b}$ follows the supermartingale process

$$u_{t-1}^{'}\left(c_{t-1}^*\right) b_{t-1} \geq \mathbf{E}_{t-1}\left[u_t^{'}(c_t^*)\, b_t\right]. \tag{9}$$

We obtain, in our general setting, the property that a bubble "never starts" in the rational expectations equilibrium (see [16]) but, as will be widely discussed later, the possibility for a bubble component to exist and burst as time goes on, cannot be excluded.

It is interesting to sketch prices evolution according to the standard Euler equation, *i.e.* as long as (1) holds with equality. In this case (4) becomes

$$u_t^{'}(c_t^*)\, p_t = \mathbf{E}_t \sum_{s=1}^{\infty} u_{t+s}^{'}\left(c_{t+s}^*\right) d_{t+s} \; + \lim_{N\to+\infty} \mathbf{E}_t\left[u_{t+N}^{'}\left(c_{t+N}^*\right) p_{t+N}\right]$$

and

$$u_t^{'}(c_t^*)\, b_t = \lim_{N\to+\infty} \mathbf{E}_t\left[u_{t+N}^{'}\left(c_{t+N}^*\right) p_{t+N}\right] \tag{10}$$

while the bubble process obeys the martingale condition

$$u_{t-1}^{'}\left(c_{t-1}^*\right) b_{t-1} = \mathbf{E}_{t-1}\left[u_t^{'}(c_t^*)\, b_t\right]$$

Clearly, in such a case the bubble component, if it exists, can never burst. Examples of this sort will be given in Section 6.

Next statement establishes fundamental values $\mathbf{f}$ to be an equilibrium, thus ensuring the sufficiency of (5) as well. Usually, this kind of results are proven by means of the familiar sufficient conditions of transversality, but, owing to the special nature of constraints, we prefer resorting to a more direct method. Details are reported in Section 8. It should be noted that we do not need the present value of future wealths $\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}(c_t^*)\, w_t$ to be finite.

Theorem 1 *Under A.1-2 and the condition* $\mathbf{E}_0\left[u_t'\left(c_t^*\right)w_t\right] < \infty$ *for all* $t \geq 0$*, a necessary and sufficient condition for an equilibrium to exist is that*

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t'\left(c_t^*\right) d_t < +\infty. \tag{11}$$

An equilibrium is given by the market fundamental values:

$$f_t = \frac{1}{u_t'\left(c_t^*\right)} \mathbf{E}_t \sum_{s=1}^{\infty} u_{t+s}'\left(c_{t+s}^*\right) d_{t+s}.$$

The theory illustrated so far is the traditional one: the fundamental value of prices is unambiguously defined by (8). As widely argued in [16], such an approach is not completely satisfactory. The main objection is that these amounts are not observable. Owing to the dynamic incompleteness of markets, there might be other valuations consistent with prices that give rise to different present values of the stream of future dividends. It is well known from the finite-horizon theory that state prices can be determined by nonexistence of opportunities for pure intertemporal arbitrage profits. In our general probabilistic structure, this can be taken into account by conveniently adopting the following terminology. Given an equilibrium price process $\mathbf{p}$, an adapted sequence $a_t(\omega)$ of strictly positive functions will be termed a *pseudo state-prices* consistent with $\mathbf{p}$, if

$$a_t p_t = \mathbf{E}_t\left[a_{t+1}\left(p_{t+1} + d_{t+1}\right)\right] \tag{12}$$

for all $t \geq 0$. Strictly speaking, the a_t's are not the traditional state-prices of Finance, because they are distorted by the probability law. However, there is a one-to-one correspondence with state prices as long as the stochastic process is given through finite information nodes. In fact, in this case (12) becomes

$$a\left(s^t\right) p\left(s^t\right) = \sum_{s^{t+1} \mid s^t} \pi\left(s^{t+1} \mid s^t\right) a\left(s^{t+1}\right) \left[p\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right]$$

where $\pi\left(s^{t+1} \mid s^t\right)$ is the transition probability and s^t, s^{t+1} are adjacent nodes (we are here using the notation in [16]). After multiplying by $\mu\left(s^t\right)$, we get

$$\overline{a}\left(s^t\right) p\left(s^t\right) = \sum_{s^{t+1} \mid s^t} \overline{a}\left(s^{t+1}\right) \left[p\left(s^{t+1}\right) + d\left(s^{t+1}\right)\right],$$

which is the traditional intertemporal no-arbitrage equation and $\overline{a}\left(s^t\right) = a\left(s^t\right)\mu\left(s^t\right)$ are the familiar state-prices. Clearly, formulation (12) suits better when the states of the world are not necessarily finite at any trading time.

The theory developed earlier can be easily embedded into this approach by using (12). For instance, we would have

$$a_0 p_0 = \mathbf{E}_0 \sum_{t=1}^{\infty} a_t d_t + \lim_{N \to \infty} \mathbf{E}_0\left(a_N p_N\right) \tag{13}$$

which might generate a different splitting between the fundamental solution and the bubble component. We will turn back to these concepts in Subsection 5.4 and in Section 6.

4 Uniqueness: sufficient conditions

In this section we present results ruling out the emergence of multiple equilibria. The first sufficient condition has been established by Kamihigashi [9]. Its proof rests on the intuition that, if a bubble occurred, an infinitely lived consumer could gain by permanently reducing his holding of the asset. To be more specific, the condition allows for a uniform downward perturbation within the feasible set without facing an infinite loss.

Theorem 2 *A sufficient condition for the fundamental price* $\mathbf{f}$ *given in (8) to be the unique equilibrium is that for some scalar* $0 < \zeta < 1$

$$\mathbf{E}_0 \sum_{t=1}^{\infty} \left[u_t\left(d_t \cdot \mathrm{e} + w_t\right) - u_t\left(\zeta d_t \cdot \mathrm{e} + w_t\right)\right] < +\infty \tag{14}$$

An useful criterion is the following corollary.

Corollary 1 *A sufficient condition for (14) is*

$$\mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}\left(\zeta d_t \cdot \mathrm{e} + w_t\right) d_t < +\infty \tag{15}$$

for some scalar $0 < \zeta < 1$.

This corollary is an immediate consequence of concavity of u_t's that entails

$$u_t\left(d_t \cdot \mathrm{e} + w_t\right) - u_t\left(\zeta d_t \cdot \mathrm{e} + w_t\right) \leq (1-\zeta) u_t^{'}\left(\zeta d_t \cdot \mathrm{e} + w_t\right) d_t \cdot \mathrm{e}$$

and thus (15) implies (14).

It is worth also noticing that, again in force of concavity,

$$u_t\left(d_t \cdot \mathrm{e} + w_t\right) - u_t\left(\zeta d_t \cdot \mathrm{e} + w_t\right) \geq (1-\zeta)\, u_t^{'}\left(c_t^*\right) d_t \cdot \mathrm{e}$$

which reveals (14) to be sufficient for (11). Therefore, Kamihigashi's condition implies existence and uniqueness of equilibria simultaneously.

A slight modification of the proof of Theorem 2 establishes a further specification of (14) focussing on a single asset i and upon the occurrence of some event $A \in \mathcal{F}_s$.

Proposition 3 *If for an event* $A \in \mathcal{F}_s$*, one has*

$$\mathbf{E}_s \sum_{t=s+1}^{\infty} \mathbf{1}_A \left[u_t\left(c_t^*\right) - u_t\left(c_t^* - (1-\zeta)\, d_t^i\right)\right] < +\infty \tag{16}$$

for some scalar $0 < \zeta < 1$*, then the bubble component, for the* i^{th} *component* p_t^i*, vanishes after epoch* s*, as long as* A *occurs. That is,* $b_t^i(\omega) = 0$*, for* $t \geq s$ *and almost all* $\omega \in A$.

A remarkable consequence of Proposition 3 is the absence of a positive bubble component for *fiat money assets*, *i.e.*, assets for which $d_t^i(\omega) \equiv 0$ for all t. In fact, in such a case, (16) is trivially true, irrespectively of agent's preferences. This generalizes Corollary 3.2 of [16] within our setting.

It is worth pointing out that sufficient conditions (14), (15) and (16), while ruling out multiple equilibria, are not necessary at all. This aspect will be discussed in the next sections where it will also seen as (14) does not preclude the existence of so-called ambiguous bubbles. However, by looking deeper into inequalities (14) and (15), the intuition on fragility of multiple equilibria becomes self-evident. In particular, the similarity between condition (11), which is necessary for the existence of at least one equilibrium, and condition (15), which rules out bubbles, clearly casts new light on the question whether price bubbles constitute a frequently verified phenomena or, on the contrary, bubble equilibria are rather a borderline event. This can be seen through two parallel arguments: either by slightly shifting away from the initial condition $y_0 = \mathrm{e}$, or through a perturbation of the dividend stream $\mathbf{d} = \{d_t\}$.

Suppose there is a bubble when the initial assets supply is $y_0 = \mathrm{v} \in \mathbf{R}_{++}^k$. Since (11) must be fulfilled, $\mathbf{E}_0 \sum_{t=1}^{\infty} u_t'(d_t \cdot \mathrm{v} + w_t)\, d_t < +\infty$. Take any initial vector $\overline{\mathrm{v}}$ such that $\overline{\mathrm{v}} \gg \mathrm{v}$. Then, there is some $\zeta < 1$ for which $\zeta\overline{\mathrm{v}} \gg \mathrm{v}$. This implies $\mathbf{E}_0 \sum_{t=1}^{\infty} u_t'(\zeta d_t \cdot \overline{\mathrm{v}} + w_t) d_t < +\infty$, which is the sufficient condition (14) for the equilibrium with initial condition $\overline{\mathrm{v}}$ to be unique. Likewise, assume that for $y_0 = \underline{\mathrm{v}} \ll \mathrm{v}$ equilibria do exist, hence $\mathbf{E}_0 \sum_{t=1}^{\infty} u_t'(d_t \cdot \underline{\mathrm{v}} + w_t) d_t < +\infty$, which in turn entails $\mathbf{E}_0 \sum_{t=1}^{\infty} u_t'(\xi d_t \cdot \underline{\mathrm{v}} + w_t) d_t < +\infty$ for all $\xi > 1$. By picking $\xi > 1$ and $\zeta < 1$ so that $\underline{\mathrm{v}} \ll \xi\underline{\mathrm{v}} \ll \zeta\mathrm{v} \ll \mathrm{v}$, $\mathbf{E}_0 \sum_{t=0}^{\infty} u_t'(\zeta d_t \cdot \mathrm{v} + w_t)\, d_t < +\infty$ must hold. But this contradicts the assumption that some bubble occur for $y_0 = \mathrm{v}$. Clearly, a similar line of reasoning applies for a perturbation of the dividends d_t.

The above arguments are summarized in the next proposition.

Theorem 3 *If for an initial endowment $y_0 = \mathrm{v} \in \mathbf{R}_{++}^k$ a price bubble occurs, then, for each initial endowment $\overline{\mathrm{v}} \gg \mathrm{v}$, there is only one equilibrium, while, for each initial endowment $\underline{\mathrm{v}} \ll \mathrm{v}$, there are no equilibria at all. Likewise, if for some dividend sequence $\mathbf{d} = \{d_t\}$ a bubble arises, then there is only one equilibrium for dividends $\{\zeta d_t\}$, with $\zeta > 1$, while, if $\zeta < 1$, there are no equilibria.*

In other words, if there are equilibria for all initial endowments in a neighborhood of v, then there is a neighborhood of v where the equilibrium is unique, that amounts to saying the set of initial endowments with bubbles has zero Lebesgue measure in $\mathbf{R}_{++}^k$.

Theorem 3 is formulated without resorting to any specification of agents' preferences, besides A.1-2. Here we present another argument in favor of bubble fragility related to agent's risk aversion. There are several conditions on utilities u_t which permit us to infer the validity of (14). We refer to Theorem 5.1 in [9] for details. We give here an application of Corollary 1 not covered by [9] and that seems rather interesting from the economic point of view.

Theorem 4 *Assume preferences u_t exhibit uniformly bounded relative risk aversion, i.e.,*

$$-\frac{u_t''(c)\, c}{u_t'(c)} \leq R \tag{17}$$

for all $c \geq 0$, $t \geq 0$ and for some scalar R. Then pricing equilibrium is uniquely determined[6]*.*

[6]It must be emphasized that twice differentiability hypothesis on preferences is not necessary at all. It is sufficient that some R exists such that $u_t'(c, \omega)c^R$ are nondecreasing for all t and for *a.e.* ω.

Proof. Let us utilize sufficient condition (15) formulated in Corollary 1. We first observe that if there is a constant $M(\zeta)$, independent of t, such that

$$u_t^{'}(\zeta c+h) \leq M(\zeta) u_t^{'}(c+h) \tag{18}$$

for some $\zeta < 1$ and for all $c \geq 0$, $h \geq 0$, $t \geq 0$, then (11) implies (15). In force of (17), the function $u_t^{'}(c+h)\, c^R$ is non-decreasing, as can be checked by calculating its derivative. Thus $u_t^{'}(\zeta c+h)\, \zeta^R \leq u_t^{'}(c+h)$ for $\zeta \leq 1$, and (18) is valid by setting $M(\zeta) = \zeta^{-R}$. ■

Theorem 4 generalizes Corollary 5.1 in [9] where CRRA utilities are treated. Some others results of Theorem 5.1 in [9] follow too from our condition (18). For instance, by assuming $u_t(c) = \beta^t u(c)$, conditions (L2) and (U2) of [9] imply (18). It is further interesting to notice that the HARA utilities, *i.e.*, those satisfying $-u''(c)/u'(c) = 1/(a+bc)$, fall into the class previously described, since (17) holds.

5 Bubbles

The main consequence of the results presented in the preceding section is that the existence of multiple equilibria is a very negligible phenomenon. Loosely speaking, Theorem 3 says that, generically, either the equilibrium is unique, or no equilibrium exists. Moreover, Theorem 4 asserts that standard preferences cannot give rise of bubbles. Nevertheless, it seems to be interesting to study the effective occurrence of positive bubbles.

In spite of the fact that furnishing an exhaustive classification is not a simple task, we try to roughly distinguish two polar cases in which the sufficient condition (14) is violated.

1. For all $\zeta < 1$,

$$\mathbf{E}_0 \sum_{t=1}^{\infty} \left[u_t(c_t^*) - u_t(\zeta d_t \cdot \mathrm{e} + w_t)\right] = +\infty$$

   with positive probability, but there is a time $N > 1$ and some constant $\zeta < 1$ such that

$$\mathbf{E}_{N-1} \sum_{t=N}^{\infty} \left[u_t(c_t^*) - u_t(\zeta d_t \cdot \mathrm{e} + w_t)\right] < +\infty.$$

2. For all time $N \geq 1$ and all $\zeta < 1$,

$$\mathbf{E}_{N-1} \sum_{t=N}^{\infty} \left[u_t(c_t^*) - u_t(\zeta d_t \cdot \mathrm{e} + w_t)\right] = +\infty$$

   with positive probability.

Obviously, models falling into the first category, exhibit prices such that $b_t = 0$ for $t \geq N-1$, and therefore bubbles, if any, must eventually burst after some time. In the following subsections we construct examples with bubbles of this type. Not surprisingly, it turns out that their occurrence is related to the violation of the Euler equation.

Models belonging to the second class having bubble component seem less dependent on violation of Euler equality. Since more elaboration is needed, they are postponed into Section 6, where two examples with a positive bubble are given, regardless of the fact that Euler equality is satisfied at each date. However, already in this section some additional material pertaining this subject will be anticipated.

5.1 Non differentiable utilities

We begin with a case where short-run utilities fail to be differentiable. It is Gilles and LeRoy's example [7], where infinitely many equilibria do exist. Price indeterminacy in this case seems to be more or less recognized. See, for instance, page 308 of [6], where, however, the prediction of indeterminacy is argued to be non-robust, owing to the fact that the utility can fail to be differentiable only on a zero Lebesgue measure set.

The economy is deterministic: $w_t = 0$ and the future dividends of a single asset are constant over time, $d_t = r$. Agents' preferences are

$$u_t(c) = \begin{cases} c - r & \text{if } c \leq r, \\ (1+r)^{-t}(c-r) & \text{if } c \geq r \end{cases}$$

Kamihigashi [9] (here we are using his notation) has already shown that price sequences $p_t = 1 + b(1+r)^t$, for $b > 0$, claimed by the original authors to be equilibria, are actually not. On the other hand, it is not difficult to construct a plethora of equilibria.

Let $a_t \in \partial u_t(r) = \left[(1+r)^{-t}, 1\right]$ be an arbitrary sequence with $\sum_{t=0}^{\infty} a_t < +\infty$. Then the price sequence

$$p_t = \frac{r}{a_t} \sum_{s=1}^{\infty} a_{t+s} \tag{19}$$

is an equilibrium. To see this, it suffices to observe that they satisfy $a_t p_t = a_{t+1}(p_{t+1} + r)$ which is the short-run optimality conditions. Straightforward modifications in the proof of Theorem 1 show that feasible consumption streams satisfy

$$\sum_{t=0}^{\infty} a_t c_t \leq r \sum_{t=0}^{\infty} a_t$$

So, thanks to concavity, $u_t(r) - u_t(c_t) \geq a_t(r - c_t)$, with a similar argument we can deduce the optimality property of equilibrium consumptions.

This is a good example to illustrate the difference between Santos and Woodford's approach and ours. We have found several equilibria, but none of them would involve a bubble component according to their theory. In fact, for any equilibrium, (19) says that these are the fundamental prices according to the state prices a_t (actually, according to any state prices, given that the market is dynamically complete). It is also to be noted that this economy does not possess equilibria other than the ones characterized by (19). This will be made clear in Section 6.

5.2 Bursting bubbles and sunspots

In this subsection we present an example of prices having a bubble of the first type, closely related to the violation of the Euler equation. It is a more general version of Example 3.2 in [9]. There are countable many states of the world, labelled by integers, *i.e.*, $\Omega = \{1, 2, 3, ...\}$. The uncertainty is completely revealed at time $t = 1$. Therefore, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and $\mathcal{F}_t = 2^{\Omega}$ for $t \geq 1$. The dividends of a single asset are $d_0 > 0$, $d_1(\omega) > 0$ and $d_t(\omega) = 0$ for $t \geq 2$. Endowments are $w_0 = w_1 = 0$ and $w_t = \overline{w} > 0$ for $t \geq 2$. Agent's preferences are given by $u_t = \beta^t v(c)$. Regarding to the utility v, it is assumed itself to satisfy the two conditions

$$\begin{aligned} \mathbf{E}\left[v'(d_1)d_1\right] &< +\infty \\ \mathbf{E}\left[v(d_1) - v(\zeta d_1)\right] &= +\infty \end{aligned} \tag{20}$$

for all $\zeta < 1$.

The fundamental values turn out to be

$$f_0 = \frac{\beta}{v'(d_0)} \mathbf{E}\left[v'(d_1)d_1\right] \qquad \text{and } f_t(\omega) = 0 \text{ for } t > 0.$$

Clearly, there are no bubbles for $t \geq 1$. This is a consequence of Proposition 3.

Let us show the existence of a positive bubble component at $t = 0$. Since $p_t = f_t = 0$ for $t \geq 1$, if we set $y_1 = 1 + \delta$, consumptions are $c_0 = d_0 - p_0\delta$, $c_1 = (1+\delta)d_1$, $c_t = \overline{w}$, for $t \geq 2$. By evaluating the objective function over this consumption plan, it is immediate to see that p_0 will be an equilibrium if the convex function

$$\varphi(\delta) = v(d_0) - v(d_0 - p_0\delta) + \beta\mathbf{E}\left[v(d_1) - v((1+\delta)d_1)\right]$$

defined over the interval $-1 \leq \delta < d_0/p_0$, achieves its minimum at $\delta = 0$. By (20), $\varphi(\delta) = +\infty$ if $\delta < 0$ and $\varphi(0) = 0$. Thanks to the convexity, the optimum lies at zero, whenever $\varphi'_+(0) \geq 0$. Simple calculations lead to $D_+\varphi(0) = v'(d_0)p_0 - \beta\mathbf{E}\left[v'(d_1)d_1\right] \geq 0$, which amounts to $p_0 \geq f_0$. Consequently, any price $p_0 \geq f_0$ is an equilibrium and the Euler equation is violated for $p_0 > f_0$. Observe that here the violation of the Euler equation is due the failure of condition (3), while (2) remains true.

To end this example, we need to specify functions v satisfying both conditions in (20). In view of Theorem 4, a good candidate turns out to be $v(c) = -e^{\frac{1}{c}}$, exhibiting unbounded relative risk aversion close to the origin. Let $d_1(n) = n^{-1}$ be the dividend payed at epoch 1 by the asset and μ_n be any probability defined over states satisfying

$$\mu_n \sim \frac{e^{-n}}{n^{2+\alpha}}$$

as $n \to \infty$, where $\alpha > 0$. The first condition of (20) becomes

$$\mathbf{E}\left[v'(d_1)d_1\right] = \mathbf{E}\left(e^{\frac{1}{d_1}}d_1^{-1}\right) = \sum_{n=1}^{\infty} e^n n\mu_n.$$

Since $e^n n\mu_n$ is asymptotically equivalent to $n^{-(1+\alpha)}$, the series converges and consequently $\mathbf{E}\left[v'(d_1)d_1\right] < +\infty$. Regarding to the second condition of (20), observe that

$$\mathbf{E}\left[v(d_1) - v(\zeta d_1)\right] = \mathbf{E}\left(e^{\frac{1}{\zeta d_1}} - e^{\frac{1}{d_1}}\right) = \sum_{n=1}^{\infty}\left(e^{\frac{n}{\zeta}} - e^n\right)\mu_n$$

where the terms of this series are asymptotically equivalent to

$$n^{-(2+\alpha)}\left[e^{n\left(\zeta^{-1}-1\right)} - 1\right]$$

which go to infinity as $n \to \infty$, and so $\mathbf{E}\left[v(d_1) - v(\zeta d_1)\right] = +\infty$.

Of course, similar examples can be easily replicated. Following the discussion in Section 4, this bubble is not robust at all. To test its fragility, set the dividend to be $d_1(n) = \overline{\zeta}n^{-1}$. Then the fundamental values are the unique equilibrium for $\overline{\zeta} > 1$, while, there are no equilibria if $\overline{\zeta} < 1$, since the first of (20) fails.

However, this example is interesting enough since a slight modification shows the possible source of sunspot bubbles.

An example of sunspot bubbles. Let the state space be $\Omega = \{L, H\} \times \mathbf{N}$, where states L and H represent extrinsic uncertainty (a sunspot) each occurring with probability 1/2, whereas the probability distribution over $\mathbf{N} = \{1, 2, ...\}$ is the same as before. The filtration is constructed by setting $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1$ be generated by the partition $\{L\} \times \mathbf{N}, \{H\} \times \mathbf{N}$, and $\mathcal{F}_t = 2^\Omega$ for $t \geq 2$. The dividends are $d_0 > 0$, $d_1(H) = d_1(L) = d_1$, $d_2(\omega) \equiv d_2(n) = n^{-1}$ and $d_t = 0$ for $t \geq 3$. Endowments are $w_0 = w_1 = w_2 = 0$, while after date $t = 3$ the agent lives thanks to an unspecified stream of endowments w_t. Suppose that all the utilities u_t are linear with the exception of u_2, which equals the function v of the previous example. We claim the existence of a continuum of equilibria given by

$$\begin{aligned} &p_0 = d_1 + (1/2)\left[p_1(H) + p_1(L)\right], \\ &p_1(H) \geq \mathbf{E}\left[v'(d_2)\, d_2\right], \\ &p_1(L) \geq \mathbf{E}\left[v'(d_2)\, d_2\right]. \end{aligned}$$

There is a sunspot equilibrium, whenever $p_1(H) \neq p_1(L)$.

To see this, denote equilibrium prices by p_0, $p_1(H)$, $p_1(L)$, $p_t = 0$, for $t \geq 2$. Necessarily, $p_0 = d_1 + (1/2)\left[p_1(H) + p_1(L)\right]$, which is the Euler equation. Denote by $y_1 = 1 + \delta_1$, $y_2(\omega) = 1 + \delta_2(\omega)$ the asset holding strategy that finances consumptions $c_0 = d_0 - p_0\delta_1$, $c_1 = (1 + \delta_1)\, d_1(\omega) + p_1(\omega)\left[\delta_1 - \delta_2(\omega)\right]$, $c_2 = \left[1 + \delta_2(\omega)\right] d_2(\omega)$ and $c_t = w_t$ for $t \geq 3$. A direct computation of the objective function for this plan leads to checking whether

$$\begin{gathered} p_1(H)\,\delta_2(H) + p_1(L)\,\delta_2(L) + \mathbf{E}\left\{v(d_2) - v\left[(1 + \delta_2(H))\, d_2\right]\right\} + \\ \mathbf{E}\left\{v(d_2) - v\left[(1 + \delta_2(L))\, d_2\right]\right\} \geq 0 \end{gathered} \tag{21}$$

holds for all pairs $\delta_2(H)$, $\delta_2(L)$. Thanks to the second of (20), we can restrict our analysis to positive values $\delta_2(H), \delta_2(L) \geq 0$. From concavity of v,

$$\mathbf{E}\left\{v(d_2) - v\left[(1 + \delta_2)\, d_2\right]\right\} \geq -\mathbf{E}\left[v'(d_2)\, d_2\right] \delta_2$$

and thus the expression in (21) is greater than

$$p_1(H)\,\delta_2(H) + p_1(L)\,\delta_2(L) - \left[\delta_2(L) + \delta_2(H)\right] \mathbf{E}\left[v'(d_2)\, d_2\right],$$

which is non-negative as long as $p_1(H) \geq \mathbf{E}\left[v'(d_2)\, d_2\right]$ and $p_1(L) \geq \mathbf{E}[v'(d_2)\, d_2]$. This proves our assert. In a different context, with two heterogeneous agents having beliefs converging to rational expectations, an example of sunspot equilibrium is given in [15]. However, the sunspot equilibrium constructed there is uniquely determined by fundamental prices, while in our example sunspot equilibria are (possibly) generated by the appearance of bubbles.

5.3 On Euler equation

To further emphasize the role of Euler strict inequality in the appearance of bursting bubbles, here it is shown that the violation of condition (14) does not imply necessarily multiple equilibria. By Proposition 3, if

$$\mathbf{E}_t \sum_{s=1}^{\infty} \left[u_{t+s}\left(c^*_{t+s}\right) - u_{t+s}\left(\zeta d_{t+s} \cdot \mathrm{e} + w_{t+s}\right)\right] < \infty,$$

then $p_s = f_s$ for all $s \geq t$. Now, if we can prove that prices p_s satisfy the Euler equation for $s = 0, 1, ..., t$, by backward induction we infer to be $p_s = f_s$ for all s. This is illustrated through the following example.

Example 1. We are using the same framework of Subsection 5.2: $\Omega = \{1, 2, ...\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t = 2^\Omega$ for $t \geq 1$, therefore the true state of the world is revealed at date $t = 1$. The utility function v and the random variable $d(n)$ are those displayed in (20). The economy is defined as follows. The preferences are $u_0 = u_1 = c$, $u_t(c) = \beta^t v(c)$ for $t \geq 2$. The dividends of one single asset are $d_0 > 0$, $d_1(n) = kv'[d(n)]\, d(n)$, $d_2(n) = d(n)$, $d_t = 0$ for $t \geq 3$ and where $k > 0$. The exogenous endowments are $w_0 = w_1 = w_2 = 0$ and $w_t = \overline{w} > 0$ for $t \geq 3$.

The equilibrium prices p_t clearly agree with their fundamental values for $t \geq 1$ and they are

$$f_1 = \beta^2 v'(d)\, d, \qquad f_t = 0, \text{ for } t > 1$$

It is easily seen that

$$\mathbf{E}_0 \sum_{t=1}^{\infty} \left[u_t(d_t + w_t) - u_t(\zeta d_t + w_t)\right] = +\infty$$

for all $\zeta < 1$ and therefore condition (14) fails. In spite of that, such a model does not exhibit bubbles since the pair of prices p_0, f_1 follows necessarily the Euler equation. To achieve this, we must check that both (2) and (3) are true. The latter is obviously valid, given that u_1 is linear. The former holds since $\beta^2 v'(d)\, d \leq Mkv'(d)\, d$. As a consequence, $p_0 = f_0$ and no bubble exists.

It is remarkable to observe that

$$\mathbf{E}_0 \left[u_1(d_1) - u_1(\zeta d_1)\right] < \infty$$

and

$$\mathbf{E}_1 \sum_{t=2}^{\infty} \left[u_t(d_t + w_t) - u_t(\zeta d_t + w_t)\right] = u_2(d_2) - u_2(\zeta d_2) < \infty,$$

since uncertainty resolves at $t = 1$. Nonetheless, by (20), the second random variable has not a finite expectation, *i.e.*,

$$\mathbf{E}_0 \mathbf{E}_1 \sum_{t=2}^{\infty} \left[u_t(d_t + w_t) - u_t(\zeta d_t + w_t)\right] = \infty.$$

Example 2. A slight modification of the last example provides a type of bubble by negating (2) of the Euler equation. This is interesting since it shows that (2) matters.

To keep things simple, we assume preferences as before, while dividends are now $d_0 > 0$, $d_1 = 0$, $d_2 = d$, $d_t = 0$ for $t \geq 3$. The exogenous endowments are $w_0 = 0$, $w_1 = \overline{w}$, $w_2 = 0$ and $w_t = \overline{w} > 0$ for all $t \geq 3$. Condition (14) is not fulfilled and (3) does. Here (2) translates into $\beta^2 v'(d)\, d \leq M\overline{w}$, which is violated as being $v'(d)\, d$ not bounded.

To check the effective existence of a bubble component for price p_0 we must follow a direct method. Denoted by y_1, $y_2(n)$ a holding strategy. Then[7], we must verify that the convex function

$$p_0(y_1 - 1) + \mathbf{E} f_1(y_2 - y_1) + \beta^2 \mathbf{E}\left[v(d) - v(y_2 d)\right]$$

[7] We skip all the details as being similar to the example in Subsection 5.2.

attains a minimum at $y_1 = y_2 = 1$, under constraints

$$0 \leq y_1 \leq 1 + d_0/p_0, \qquad 0 < y_2 \leq y_1 + \overline{w}/f_1$$

with $f_1 = \beta^2 v'(d)\, d$. By using again specifications $v(c) = -e^{1/c}$ and $d(n) = n^{-1}$, it follows that, by choosing $y_1 < 1$, the amount $y_1 + \overline{w}/f_1$ is definitively less than $1 - \varepsilon$ and therefore $y_2(n)$ is definitively less than $1 - \varepsilon$. This entails

$$\mathbf{E}\left[v(d) - v(y_2 d)\right] = +\infty$$

and we conclude that we can restrict our attention to assets holding with $y_1 \geq 1$. By adopting standard techniques of convex analysis it is easily seen that our claim is true provided that $p_0 \geq \beta^2 \mathbf{E}\left[v'(d)\, d\right] = f_0$.

5.4 Ambiguous bubbles

The unique example in [16] of bubbles for Lucas' models deserves a more thoroughly discussion since it is related to bubbles originated by different valuations of the stream of future dividends. Here an example which generalizes the binomial Example 4.5 of [16] is illustrated.

Assume agent's preferences are $\sum_{t=0}^{\infty} \beta_t(\omega)\, u(c_t)$, with u being any concave function and where the stochastic discount factors satisfy $\mathbf{E}_0 \sum_{t=1}^{\infty} \beta_t(\omega) < \infty$. There is a single asset paying a constant dividend forever, *i.e.*, $d_t \equiv 1$, while the endowments of the consumption good are $w_t \equiv 0$. Under the previous condition, the economy has an equilibrium given by fundamental values

$$f_t = \frac{1}{\beta_t} \mathbf{E}_t \sum_{s=1}^{\infty} \beta_{t+s}$$

Moreover, condition (14) translates into

$$\mathbf{E}_0 \sum_{t=1}^{\infty} \beta_t \left[u(1) - u(\zeta)\right] < \infty$$

that ensures the above equilibrium to be the unique one.

The probabilistic space $(\Omega, \mathbf{F}, \mu)$ underlying the economy is such that a sequence of events $\{A_t\}_{t=0}^{\infty}$ exists having the following properties: $A_t \supset A_{t+1}$, $A_0 = \Omega$, $A_t \in \mathcal{F}_t$ and $\mathbf{E}_t\left(\mathbf{1}_{A_{t+1}}\right) = q\mathbf{1}_{A_t}$, for all t and with $0 < q < 1$. Note that this implies that $\mathbf{E}_t\left(\mathbf{1}_{A_{t+T}}\right) = q^T \mathbf{1}_{A_t}$ and $\mu(A_t) \to 0$ as $t \to \infty$. With this sequence of events, we shall construct the discount factor process in the following manner:

$$\beta_t(\omega) = \delta^t (1+r)^{-t} \quad \text{if } \omega \in A_t$$
$$\beta_t(\omega) = \delta^i \alpha_i (1+r)^{-t} \quad \text{if } \omega \in A_i \quad \text{but } \omega \notin A_{i+1}$$

Here $0 < \delta < 1$, $r > 0$ and scalars α_i are defined as

$$\alpha_i = (1 - \delta q)(1 - q)^{-1} + (1 + r)^i r (1 - \delta) \Delta (1 - q)^{-1}$$

with $\Delta > 0$. Through lengthy and tedious calculations, it can be shown that the unique price process is given by

$$f_t = r^{-1} + (1 + r)^t q^{-t} \Delta \mathbf{1}_{A_t}$$

Of course, such prices obey the Euler equation

$$\beta_t f_t = \mathbf{E}_t \left[\beta_{t+1} \left(f_{t+1} + 1 \right) \right].$$

However, due to markets dynamic incompleteness, this is not necessarily the unique intertemporal no-arbitrage condition equally consistent with prices f_t. More explicitly, there might exist a pseudo state-price process $a_t(\omega)$ such that

$$a_t f_t = \mathbf{E}_t \left[a_{t+1} \left(f_{t+1} + 1 \right) \right].$$

Actually this is the case if one chooses the state-price process $a_t = (1+r)^{-t}$, which leads to

$$f_t = (1+r)^{-1} \mathbf{E}_t \left(f_{t+1} + 1 \right)$$

as one can easily verify. In terms of the state prices $a_t = (1+r)^{-t}$, the fundamental value of the asset turns out to be $1/r$, so that along the set A_t, one observes a bubble component amounting to $(1+r)^t q^{-t} \Delta$. Note that this bubble is bursting almost surely.

This analytical exercise proves that condition (14), while ruling out multiple equilibria, does not preclude potential valuation bubbles due to different choices of state prices. This example will be recovered further accordingly to the theory developed in the final section, where another example of ambiguously defined bubbles of a different nature will be furnished.

6 Transversality Condition and Bubbles

In this final section we fill two gaps met so far. First, as it has been already shown in Subsection 5.3, uniqueness condition (14) is far from being necessary. For instance, as it will be shown in short, in deterministic models uniqueness of the pricing equilibrium is true regardless of condition (14), thus addressing the question on to what extent a single equilibrium can be inferred without resorting to (14). Second, we shall provide conditions which preclude the occurrence of ambiguous bubbles of the kind discussed in Subsection 5.4. Specifically, we want to rule out bubbles generated by adopting a generic $\mathbf{F}$-adapted pseudo state-prices $a_t(\omega)$ consistent with the equilibrium, that is, satisfying (12). Since, in particular, we can choose $a_t = u'(c_t^*)$, throughout this section we shall always make the assumption that prices satisfy the Euler equation.

It will be proposed the transversality condition

$$\sum_{t=0}^{\infty} \frac{d_t^i}{p_t^i} = +\infty \tag{22}$$

defined for each fixed asset i. To illustrate what we have in mind, it is useful to start with the one-asset model without uncertainty, where (22) has a sharp explanation. In this section we shall make a wide usage of the following well-known inequalities

$$1 + \sum_{t=0}^{\infty} \alpha_t \leq \prod_{t=0}^{\infty} (1 + \alpha_t) \leq \exp\left(\sum_{t=0}^{\infty} \alpha_t \right) \tag{23}$$

being true for all sequences of scalars $\alpha_t \geq 0$. Thanks to (23), the series $\sum_{t=0}^{\infty} \alpha_t$ diverges if and only if $\prod_{t=0}^{\infty} (1 + \alpha_t)$ diverges as well.

6.1 Deterministic Economies

Let then $\{p_t\}$ be a price sequence satisfying the Euler equation, for a single asset in a world with no uncertainty. Then, $u'_{t-1}(c_{t-1})\,p_{t-1} = u'_t(c_t)(d_t + p_t)$, where d_t is the stream of future dividends. If we introduce the infinite product $\prod_{t=1}^{\infty}(1 + d_t/p_t)$, straightforward computations yield

$$\prod_{t=1}^{\infty}\left(1+\frac{d_t}{p_t}\right) = \prod_{t=1}^{\infty}\left[1+\frac{u'_t(c_t)\,d_t}{u'_t(c_t)\,p_t}\right] = \prod_{t=1}^{\infty}\frac{u'_t(c_t)(d_t+p_t)}{u'_t(c_t)\,p_t}$$

$$= \prod_{t=1}^{\infty}\left[\frac{u'_{t-1}(c_{t-1})\,p_{t-1}}{u'_t(c_t)\,p_t}\right] = \lim_{t\to\infty}\frac{u'_0(c_0)p_0}{u'_t(c_t)\,p_t} = \frac{p_0}{b_0}$$

where the last is true because necessarily $\lim_{t\to\infty} u'_t(c_t)\,p_t = u'_0(c_0)\,b_0$ (see (10)). Therefore, we infer

$$\prod_{t=1}^{\infty}\left(1+\frac{d_t}{p_t}\right) = \frac{p_0}{b_0}$$

Clearly, by virtue of (23) a positive bubble will exist if and only if the value of series $\sum_{t=1}^{\infty} d_t/p_t$ is finite. We have thus established the equivalence

$$\sum_{t=1}^{\infty} d_t/p_t = \infty \Longleftrightarrow \text{no bubbles} \tag{24}$$

On the other hand, we shall prove at the end of this section that the condition $\sum_{t=1}^{\infty} d_t/p_t = \infty$ is necessary in order that p_t be an equilibrium. Therefore, we can state the other equivalence

$$\sum_{t=1}^{\infty} d_t/p_t = \infty \Longleftrightarrow \{p_t\} \text{ is an equilibrium} \tag{25}$$

Coupling (24) and (25) we infer that valuation bubbles are necessarily absent, at least in the deterministic model. Of course all this arguments are already well know (see for example [8]) and are intimately related to the exclusion of Ponzi schemes as feasible strategies. Since the asymptotic behavior of series $\sum_{t=1}^{\infty} d_t/p_t$ is the same as that of the infinite product $\prod_{t=1}^{\infty}(1 + d_t/p_t)$, let us focus on the latter, which allows for two interpretations. Consider the self-financing asset holding strategy $\widehat{y}_t$ in which all the dividends payed by the asset are invested and so the agent consume his exogenous income at any trading date. Its time evolution is $\widehat{y}_t = \prod_{n=0}^{t-1}(1 + d_n/p_n)\,\widehat{y}_0$ and, therefore, the above discussion leads to the conclusion that the asset grows boundless whenever prices do not involve a bubble. Another slightly different interpretation is through Ponzi strategies' viewpoint. Suppose there is a short-selling, $y_1 < 0$, at date 1 and the debt is rolled-over at every trading date $t > 1$. The debt accumulated at time t will be $y_t = \prod_{n=1}^{t-1}(1 + d_n/p_n)\,y_1$ and consequently the Ponzi strategy turns out to be unbounded as long as the bubble is absent.

6.2 Stochastic Economies

The extension of the foregoing argument to our general setting, requires to solve a certain number of technical difficulties arising for the presence of many assets and uncertainty. We need to construct self-financing asset holding strategy separately for each single asset $i \in \{1, 2, ..., k\}$. A disturbing event may happen: for certain states of nature and at some date, some prices could vanish. Since we want to cope with a theory comprehensive enough to tackle this situation too, we deal with zero prices as follows.

For each single asset $i \in \{1, 2, ..., k\}$, consider the self-financing plan $\{\widehat{y}_t(i)\}$ such that

$$\widehat{y}_t^j(i) = 0, \text{ for all } t \geq 0 \text{ and } j \neq i.$$

Following this plan, at each date t, dividends produced by the i^{th} asset are re-invested in the same asset i. We shall identify the i^{th} component of vector $\widehat{y}_t(i)$ with the plan itself and, to ease notation, we shall write $\widehat{y}(i) \equiv \widehat{y}_t^i$. By assuming $\widehat{y}_0^i$ to be a strictly positive scalar, and p_t^i to be strictly positive, the plan $\widehat{\mathbf{y}}^i = \{\widehat{y}_t^i\}$ is determined recursively as

$$\widehat{y}_{t+1}^i = \left(1 + d_t^i / p_t^i\right) \widehat{y}_t^i. \tag{26}$$

In order to let (26) be defined as well for prices not ever strictly positive, we shall adopt the convention $\widehat{y}_{t+1}^i(\omega) = +\infty$, whenever $p_t^i(\omega) = 0$. Define the sets[8] $P_t^i = \{p_t^i = 0\}$, in force of the Euler equation one has $P_t^i \subseteq P_{t+1}^i$, for all $t \geq 0$. Consequently, $p_s^i = 0$ for all $s > t$ and all $\omega \in P_t^i$. We have thus constructed an extended real-valued increasing sequence $\{\widehat{y}_t^i\}$.

Setting $\widehat{y}_\infty^i = \lim_{t \to \infty} \widehat{y}_t^i$, from (26) and by means of our convention for d_t^i / p_t^i, clearly $\widehat{y}_t^i \uparrow \widehat{y}_\infty^i$. Specifically,

$$\widehat{y}_\infty^i = \prod_{t=0}^{\infty} \left(1 + d_t^i / p_t^i\right) \widehat{y}_0^i. \tag{27}$$

Proposition 4 *Let $\{p_t, a_t\}$ be two $\mathbf{F}$-adapted processes satisfying (12). Consider the self-financing plan $\widehat{\mathbf{y}}^i$ generated by (26). If we agree upon setting $p_t^i(\omega) \widehat{y}_{t+1}^i(\omega) = 0$ when $p_t^i(\omega) = 0$, then the process $a_t p_t^i \widehat{y}_{t+1}^i$ is a supermartingale.*

Proof. Denote by C_t^i the complement of set $P_t^i = \{p_t^i = 0\}$. By definition,

$$\mathbf{E}_{t-1}\left(a_t p_t^i \widehat{y}_{t+1}^i\right) = \mathbf{E}_{t-1}\left(\mathbf{1}_{C_t^i} a_t p_t^i \widehat{y}_{t+1}^i\right).$$

According to (26)

$$\mathbf{1}_{C_t^i}(p_t^i \widehat{y}_{t+1}^i) = \mathbf{1}_{C_t^i}(p_t^i + d_t^i) \widehat{y}_t^i$$

that, by means of (12), leads to

$$\mathbf{E}_{t-1}\left(a_t p_t^i \widehat{y}_{t+1}^i\right) = \mathbf{E}_{t-1}\left[a_t \mathbf{1}_{C_t^i}(p_t^i + d_t^i) \widehat{y}_t^i\right] \leq a_{t-1} p_{t-1}^i \widehat{y}_t^i$$

as was to be shown. ■

The first important result is the next statement which requires the value of series (22) to hold uniformly. Of course, this means that for any number N, there exists some time t_0 so that $\sum_{t=0}^{T} d_t^i / p_t^i \geq N$ *a.s.* for all $T \geq t_0$. In accordance with (23), we shall have $\widehat{y}_t^i \uparrow +\infty$ uniformly as well.

[8] It should be noted that sets P_t^i do not depend on the choice of state-prices a_t employed.

Proposition 5 *If the prices process* $\mathbf{p}$ *obeys the Euler equation and, in addition, (22) holds uniformly, then* $p_t^i = f_t^i$ *for all* t*, that is, the* i^{th} *component of the bubble is zero. In addition, prices* p_t^i *unambiguously involve no bubble.*

Proof. Let us fix any pseudo state-prices consistent with $\mathbf{p}$. We know from Proposition 4 that $a_t p_t^i \widehat{y}_{t+1}^i$ is a supermartingale. Hence, $\mathbf{E}_0\left(a_t p_t^i \widehat{y}_{t+1}^i\right) \leq a_0 p_0^i \widehat{y}_1^i$. This is equivalent to $\mathbf{E}_0\left(\mathbf{1}_{C_t^i} a_t p_t^i \widehat{y}_{t+1}^i\right) \leq a_0 p_0^i \widehat{y}_1^i$, where the set C_t^i is the same as in the proof of Proposition 4. Since the sequence $\{\widehat{y}_t^i\}$ diverges uniformly, for any N we can find a time T so that $\widehat{y}_{t+1}^i \geq N$ for all $t \geq T$. Hence, $\mathbf{E}_0\left(\mathbf{1}_{C_t^i} a_t p_t^i\right) \leq N^{-1} a_0 p_0^i \widehat{y}_1^i$. On the other hand, $\mathbf{E}_0\left(\mathbf{1}_{P_t^i} a_t p_t^i\right) = 0$, which gives $\mathbf{E}_0\left(a_t p_t^i\right) \leq N^{-1} a_0 p_0^i \widehat{y}_1^i$ and, in turn, $\mathbf{E}_0\left(a_t p_t^i\right) \to 0$ as $t \to \infty$. Now, in view of (13), we can infer that the bubble component, relatively to the selected state-prices, vanishes. ■

Proposition 5 deserves a series of comments. It establishes only the right-hand arrow in both (24) and (25) for the stochastic framework, and it requires uniform divergence of the series (22). Subsection 5.4 provides a counterexample stressing that uniform divergence is needed. There, ambiguous bubbles exist even if the series (22) goes to infinity almost surely; however, of course, the limit does not hold uniformly. Another example is given at the end of this subsection. Clearly, the uniform divergence of the series acts as a transversality condition at infinity: the short-run condition expressed through the Euler equation plus (22) imply the optimality of the allocation. Unfortunately, it is hard to find conditions that establish the opposite implication in (24) and (25) for the stochastic setting.

In order to exploit the full strength of (22) in ruling out bubbles, we need to find conditions ensuring (22) to hold uniformly. In the following, we consider restrictions either on discounting properties of preferences or on the stochastic process governing the states of the world.

Next assumption establishes a condition on preferences closely related to the property of agent's impatience, relatively to a fixed single asset i.

A. 3 *There is a non-negative scalar sequence* $\{\sigma_t\}$*, having the following properties:*

i)

$$\sum_{t=1}^{\infty} \sigma_t = +\infty$$

ii) *for each integer* s *and all* $A \in \mathcal{F}_s$ *with* $\mu(A) > 0$*, there exists a scalar* $\zeta = \zeta(s, A)$*, with* $0 < \zeta < \sigma_s^{-1}$*, depending on* s *and* A *and such that the consumption stream* $\widetilde{\mathbf{c}} = \{\widetilde{c}_t\}$*, defined by*

$$\widetilde{c}_t = \begin{cases} c_t^*, \text{ for } 0 \leq t \leq s-1, \\ c_t^* + \zeta d_t^i, \text{ for } t = s, \\ c_t^* - \zeta \sigma_s d_t^i, \text{ for } t \geq s+1 \end{cases}$$

overtakes $\mathbf{c}^*$ *over* A*. That is:*

$$\begin{array}{ll} \liminf_{N \to +\infty} & \mathbf{E}_0 \sum_{t=0}^{N-1} \mathbf{1}_A \left[u_t\left(\widetilde{c}_t\right) - u_t\left(c_t^*\right)\right] = \\ & \mathbf{E}_0 \sum_{t=0}^{\infty} \mathbf{1}_A \left[u_t\left(\widetilde{c}_t\right) - u_t\left(c_t^*\right)\right] > 0. \end{array} \tag{28}$$

A.3 can be viewed as a weaker variant of that postulated by Santos and Woodford [16] (where they set $\sigma_t \equiv \sigma > 0$ and $\zeta \equiv 1$) as well as the uniform lower bound on impatience assumption in Magill and Quinzii [13]. Note also that A.3 is more restrictive than condition (14). In fact, from (i) it follows that a time N exists so that $\sigma_N > 0$. If we set $A = \Omega$ in (ii), we get easily

$$\mathbf{E}_0 \sum_{t \geq N+1} \left[u_t\left(c_t^*\right) - u_t\left(c_t^* - \zeta \sigma_N d_t^i\right)\right] < \mathbf{E}_0 \left[u_N\left(c_N^* + \zeta d_N^i\right) - u_N\left(c_N^*\right)\right] < \infty$$

where the last inequality is true in force of (ii) of Definition 1. Therefore, (16) is valid at least for $t \geq N+1$. If on one hand A.3 is more restrictive than (14), on the other hand it guarantees our desired asymptotic property of the series, as next proposition establishes.

Theorem 5 *If* $\mathbf{p}$ *is an equilibrium and A.3 holds true, then*

$$\sum_{t=0}^{\infty} \frac{d_t^i}{p_t^i} = +\infty \tag{29}$$

holds uniformly. Consequently, the i^{th} *bubble component is zero.*

The next statement derives the asymptotic behavior of series (29) without resorting on any hypothesis on agent's preference. Specifically, time-separability of preferences is not required; the only property needed is their strict monotonicity with respect to each t-period consumption. Unfortunately, the following strong assumption on the probabilistic structure is needed.

A. 4 *The probability space* $(\Omega, \mathbf{F}, \mu)$ *enjoys the following property: for all sequences* $A_t \downarrow A$, *with* $A_t \in \mathcal{F}_t$ *and* $\mu(A) > 0$, *there are a date* s *and a set* $B \subset A$ *such that* $\mu(B) > 0$ *and* $B \in \mathcal{F}_s$.

Theorem 6 *If the uncertainty is described by a probability space satisfying A.4, then (29) is true a.s.*

With this result at hand, it is immediately seen that the left-hand implications in (24) and (25) are true for the deterministic model, where pointwise and uniform divergence are equivalent. Moreover, it is possible to construct models with a dividend process such that almost surely divergence imply uniform divergence.

We illustrate the foregoing discussion by means of the following examples.

Example 1. Clearly, the example with ambiguously defined bubbles, illustrated in Subsection 5.4, violates condition A.3. Still, it is interesting to show this directly to see how our formulation of A.3 is perhaps the tightest one. It suffices to pick in A.3 the sequence of events A_t utilized in the construction of the example in Section 5.4. Arguing by contradiction and accordingly to (28), there would be

$$\mathbf{E}_0 \sum_{t=s}^{\infty} \mathbf{1}_{A_s}\left[u_t\left(\widetilde{c}_t\right) - u_t\left(c_t^*\right)\right] > 0$$

After calculations that we skip here, one gets to

$$u\left(1+\zeta_s\right) - u\left(1\right) > r^{-1}\left[u\left(1\right) - u\left(1-\zeta_s\sigma_s\right)\right]\left[1 + \left(1+r\right)^s q^{-s}\Delta\right]$$

for some two scalar sequences $\{\zeta_t\}$ and $\{\sigma_t\}$. On the other hand, thanks to concavity, we have

$$u(1) - u(1 - \zeta_s \sigma_s) \geq \sigma_s [u(1 + \zeta_s) - u(1)]$$

which, inserted in the previous one, leads to

$$\sigma_s \leq r \left[1 + (1 + r)^s q^{-s} \Delta\right]^{-1}$$

thus contradicting the fact that $\sum_{t=1}^{\infty} \sigma_t = \infty$.

Example 2: Ambiguous bubbles and Petersburg assets. Despite of having labelled Assumption A.3 as a condition related to the property of impatience of agents, as a matter of fact it involves a more complicated interplay between the discounting and the nature of dividend's distribution, as it is shown in the present example.

Let $\Omega = \{1, 2, ...\}$ and the σ-algebra $\mathcal{F}_t$ be generated by the finite partition of sets $\{1\}$, $\{2\}, ..., \{t\}$, $\{t+1, t+2, ...\}$. The probability measure assigned over Ω is $\mu(n) > 0$ for all n. Since all sequences $A_t \downarrow A$ with $A \notin \mathcal{F}_s$ are such that $\mu(A) = 0$, A.4 holds true and consequently, by Theorem 6, (29) holds almost surely.

If we set the dividends of a single asset to be $d_t(\omega) > 0$ for $\omega = t$ and $d_t(\omega) = 0$ otherwise, it is easy to realize that for any price equilibrium one has $p_t(\omega) = 0$ for all $\omega \leq t$, while $p_t(\omega) > 0$ for $\omega \geq t+1$, no matter whatever preferences and exogenous resources are given. The series certainly goes to infinity, however the limit is never uniform across states since

$$\mu\left(\sum_{t=1}^{T} \frac{d_t}{p_t} \geq N\right) = \sum_{n=1}^{T} \mu(n)$$

for all $N > 0$. Therefore, in view of Theorem 5, assumption A.3 fails. This may even be checked directly by looking at its definition: the dividends vanish over events with positive probability. Actually we are able to demonstrate the existence of valuation bubbles.

Given that the informational structure is finite at any trading date, it is simpler to describe it by means of a tree. Among the $t+1$ information sets of $\mathcal{F}_t$, we name $s^t = \{t+1, t+2, ...\}$ and $m^t = \{t\}$. The remaining nodes will be little relevant and thus we do not assign them any particular symbol. With this notation at hand, all nodes s^t have two immediate successors s^{t+1} and m^{t+1}, while all others nodes have only one immediate successor. According to this notation, we have $d(m^t) > 0$ and $d(\cdot) = 0$ elsewhere, while $p(s^t) > 0$ and $p(\cdot) = 0$ elsewhere.

We are now planning to assign a state price process $a(\cdot)$ consistent with the given equilibrium $p(\cdot)$. It is sufficient to specify the sequence $a(s^t)$ since elsewhere the state prices can be any. Let us fix a number $0 \leq \Delta < p(s^0)$ and a sequence $a(m^t)$ such that $\sum_{t=1}^{\infty} a(m^t) d(m^t) < \infty$. If we define

$$a(s^t) = p(s^t)^{-1} \left[\Delta \left[p(s^0) - \Delta\right]^{-1} \sum_{s=1}^{\infty} a(m^s) d(m^s) + \sum_{s=t+1}^{\infty} a(m^s) d(m^s)\right]$$

it is readily seen that such state prices are consistent with the equilibrium since

$$a(s^t) p(s^t) = a(s^{t+1}) p(s^{t+1}) + a(m^{t+1}) d(m^{t+1})$$

and, moreover,

$$\lim_{t \to \infty} \frac{a(s^t) p(s^t)}{a(s^0)} = \Delta,$$

which proves the existence of a positive bubble along the states s^t. Clearly the bubble is ambiguous since it disappears by setting $\Delta = 0$. In this case, the prices agree with the fundamental value according to a certain state price process (of course this is also a consequence of Theorem 3.1 in [16]).

If one sets the probability measure $\mu(n) = (1-q)^{n-1} q$, the random process can be implemented by tossing an unfair coin (with probability q of heads) until heads comes up. Thus the model in our example resembles the Petersburg game.

Example 3. Whenever the sample space Ω has finitely many elements, Property A.4 is trivially fulfilled and, clearly, (22) holds uniformly. Accordingly, there are no bubbles. If utilities are differentiable, this proves the uniqueness of equilibrium and, in turn, the equivalence (25) displayed at the beginning for deterministic models. Whenever the preferences are not differentiable, as in Subsection 5.1, it follows that there are no ambiguous bubbles.

The theory developed in this section provides for another argument that clarifies how sufficient condition (14) is not necessary at all. Indeed, it is easy to construct examples that violate (14) and at the same time uniqueness of equilibrium is assured by Theorem 6. Consider $u_t(c) = -t^{-2-\alpha}c^{-t}$, for $t > 0$ and $\alpha > 0$. This family displays unbounded relative risk aversion since u_t's relative risk-aversion index equals $t+1$. Suppose $w_t = 0$ and the dividend is steadily $d_t = 1$. It is readily checked that

$$\sum_{t=1}^{\infty} u_t^{'}(d_t)\, d_t = \sum_{t=1}^{\infty} t^{-1-\alpha} < \infty$$

and thus the asset can be priced by its fundamental value. On the other hand,

$$\sum_{t=1}^{\infty} [u_t(d_t) - u_t(\zeta d_t)] = \sum_{t=1}^{\infty} t^{-2-\alpha} \left(\zeta^{-t} - 1\right) = \infty,$$

for all $\zeta < 1$, consequently sufficient condition for uniqueness (14) is violated.

6.3 Truly Infinite Economies

In this final Subsection we present two non-trivial and truly infinite economies exhibiting bubbles. In both of them the states of the world are countable and are constructed by means of Petersburg assets. It is worth noticing that assumption A.4, and thus Theorem 6, holds in both cases.

Example 4: infinite states at each epoch. This example incorporates a Petersburg asset into the model described in Subsection 5.2 and therefore countably many states of the world are observable at each trading date. The relevant nodes are s^t, $t = 0, 1, ...$ where s^0 is the initial node, and m_n^t, with $t = 1, 2, \ldots$ and $n = 1, 2, \ldots$. Each node s^t has a countably many immediate successors s^{t+1}, $m_1^{t+1}, m_2^{t+1}, m_3^{t+1}, \ldots$. Every other node has only one immediate successor. Such nodes will not labelled since non-relevant. The stationary transition probabilities are

$$\pi\left(m_n^{t+1} \mid s^t\right) = \mu_n > 0;$$
$$\pi\left(s^{t+1} \mid s^t\right) = \mu_0 > 0$$

where $\sum_{n=0}^{\infty} \mu_n = 1$ and μ_n is the same as that used in Subsection 5.2, that is, $\mu_n \sim e^{-n}/n^{2+\alpha}$ as $n \to \infty$, $\alpha > 0$.

Preferences are described by $u(m_n^t, c) = \beta^t v(c)$ with v satisfying (20) of Subsection 5.2. Along every other node, the preferences are linear. Specifically, $u(s^t, c) = \beta^t c$. The asset dividends are $d(m_n^t) = n^{-1}$ and $d = 0$ elsewhere. At each date t, endowments are $w(m_n^t) = 0$ for all n, while at every other node the agent receives a fixed amount of consumption good $\overline{w} > 0$. Specifically, $w(s^t) = \overline{w}$.

Again $p(m_n^t) = 0$ and $p(s^t) > 0$. The Euler equation turns out to be

$$p\left(s^t\right) = \beta M + \beta\mu_0 p\left(s^{t+1}\right) \tag{30}$$

having denoted $M = \sum_{n=1}^{\infty} v'\left(n^{-1}\right) n^{-1}\mu_n$. The fundamental values

$$f\left(s^t\right) = \left(1 - \beta\mu_0\right)^{-1}\beta M = f$$

are stationary along states s^t.

Let us evaluate the objective function for a feasible consumption stream. Let $y_{t+1} \equiv y(s^t)$ be a generic asset holding strategy, we get

$$\mathbf{E}_0 \sum_{t=0}^{N} \left[u_t\left(c_t^*\right) - u_t\left(c_t\right)\right] = \begin{array}{l} \sum_{t=1}^{N} \beta^t \mu_0^{t-1} \sum_{n=1}^{\infty} \mu_n \left[v\left(n^{-1}\right) - v\left(n^{-1} y_t\right)\right] \\ + \sum_{t=0}^{N} \beta^t \mu_0^t p\left(s^t\right)\left(y_{t+1} - y_t\right) \end{array} \tag{31}$$

It is immediately seen from (31) that only values $y_t \geq 1$ for all t are relevant, otherwise, by (20), the first term in the right-hand side of (31) is $+\infty$ for some N. By keeping this restriction and exploiting the estimate,

$$v\left(n^{-1}\right) - v\left(n^{-1} y_t\right) \geq v'\left(n^{-1}\right) n^{-1}\left(1 - y_t\right)$$

we have that the right-hand side of (31) is greater than

$$\sum_{t=1}^{N} \beta^t \mu_0^{t-1} M\left(1 - y_t\right) + \sum_{t=0}^{N} \beta^t \mu_0^t p\left(s^t\right)\left(y_{t+1} - y_t\right)$$

Rearranging terms we can write

$$\sum_{t=1}^{N} \beta^{t-1}\mu_0^{t-1}\left[p\left(s^{t-1}\right) - \beta M - \beta\mu_0 p\left(s^t\right)\right]\left(y_t - 1\right) + \beta^N \mu_0^N p\left(s^N\right)\left(y_{N+1} - 1\right)$$

which is non-negative as long as

$$p\left(s^{t-1}\right) \geq \beta M + \beta\mu_0 p\left(s^t\right).$$

This is the Euler inequality associated with (30). Hence, any price sequence following the previous inequality is an equilibrium.

It is worth writing down explicitly such dynamic of prices in order to appreciate what kind of bubbles may arise. Let $\gamma_t \geq 0$ be a sequence of numbers such that

$$p\left(s^{t-1}\right) = \beta M + \beta\mu_0 p\left(s^t\right) + \gamma_{t-1}$$

The addendum γ_{t-1} measures to what extent the Euler equation is being violated. By assuming $\sum_{t=0}^{\infty} \left(\beta\mu_0\right)^t \gamma_t < \infty$, the solutions to the Euler inequality are given by

$$p\left(s^t\right) = f + b^e\left(s^t\right) + b^a\left(s^t\right)$$

with

$$b^e(s^t) = \sum_{r=0}^{\infty} (\beta\mu_0)^r \gamma_{t+r},$$
$$b^a(s^{t+1}) = (\beta\mu_0)^{-1} b^a(s^t)$$

There are two bubble components. If we set $b^a(s^0) = 0$, there is no asymptotic bubble, but the Euler equation may be continuously violated, whenever $\gamma_t > 0$ for all t. On the other hand, if we set $\gamma_t = 0$ for all t, the prices obey to the Euler equation but there may be a bubble growing exponentially along states s^t.

Example 5: finite states at each epoch. Here, again, we make use of a Petersburg asset, but now, finitely many information nodes are observable at every trading date. This construction seems to be meaningful since no violation of the Euler equation may occur. Nonetheless, even with Euler equality, multiple equilibria may arise by modeling agents having an increasing relative risk-aversion over time.

The initial node is s^0. The t-time relevant nodes are s^t, $t \geq 0$ and m^t, $t \geq 1$. Each node s^t has two immediate successors s^{t+1} and m^{t+1}, while every other node has only one successor, that will not be labelled. The transition probabilities are uniform:

$$\pi(m^{t+1} \mid s^t) = \pi(s^{t+1} \mid s^t) = 1/2$$

and agents' preferences are

$$u(m^t, c) = v_t(c) = -2^t t^{-2-\alpha} c^{-t}$$

with $\alpha > 0$, and linear elsewhere. More specifically, $u(s^t, c) = \beta^t c$ with $0 < \beta < 1$. Like in Example 3, preferences exhibit increasing relative risk aversion along states m^t. The dividends payed by a single asset are $d(m^t) = 1$, for all $t \geq 1$, and 0 elsewhere. The endowments are $w(m^t) = 0$ and $w = \overline{w} > 0$ on every other node.

While an argument similar to that in Example 3 shows that condition (11) holds, condition (14) fails, because

$$\mathbf{E}_0 \sum_{t=1}^{\infty} [v_t(1) - v_t(\zeta)] = \sum_{t=1}^{\infty} t^{-2-\alpha} (\zeta^{-t} - 1) = \infty.$$

Clearly prices must vanish outside states s^t. For prices along states s^t, we can easily calculate the Euler equation. Denote $p_t \equiv p(s^t)$, then

$$p_t = (1/2)\left[\beta p_{t+1} + \beta^{-t} v'_{t+1}(1)\right]. \tag{32}$$

By iterating it, we get

$$p_t = \beta^{-t} \sum_{s=1}^{\infty} 2^{-s} v'_{t+s}(1) + \lim_{n\to\infty} 2^{-n} \beta^n p_{t+n}$$

where the first addendum is the fundamental value and the second is the bubble component. This series converges, as it is easy to check, and therefore the bubble obeys to the martingale law

$b_{t+1} = 2\beta^{-1}b_t$ along states s^t. In other words, the bubble component can never burst along states s^t.

Now we prove that these prices with positive bubbles are consistent with equilibrium requirements. Denote by $y_{t+1} \equiv y(s^t)$ a feasible trading plan. Thus, consumption is given by

$$\begin{aligned} c\left(s^t\right) &= p_t\left(y_t - y_{t+1}\right) + \overline{w}, \\ c\left(m^t\right) &= y_t \end{aligned}$$

and

$$\mathbf{E}_0 \sum_{t=0}^{N} \left[u_t\left(c_t^*\right) - u_t\left(c_t\right)\right] = \begin{array}{l} \sum_{t=1}^{N} 2^{-t}\left[v_t(1) - v\left(y_t\right)\right] + \\ \sum_{t=0}^{N} 2^{-t}\beta^t p_t\left(y_{t+1} - y_t\right) \end{array} \tag{33}$$

The proof will be accomplished by considering strategies y_t separately in the following exhaustive classes: a) $\liminf_{t\to\infty} y_t \geq 1$; b) $\liminf_{t\to\infty} y_t < 1$.

Through the same method adopted in Example 4 and using the inequality

$$v_t(1) - v_t\left(y_t\right) \geq v_t'(1)\left(1 - y_t\right)$$

the right-hand side of (33)is greater than

$$\begin{aligned} &\sum_{t=1}^{N} 2^{-t+1}\beta^{t-1}\left[p_{t-1} - 2^{-1}\beta^{1-t}v_t'(1) - 2^{-1}\beta p_t\right]\left(y_t - 1\right) \\ &\qquad + 2^{-N}\beta^N p_N\left(y_{N+1} - 1\right) \\ &= 2^{-N}\beta^N p_N\left(y_{N+1} - 1\right), \end{aligned}$$

where equality holds thanks to (32). Since $2^{-N}\beta^N p_N \to b_0 \geq 0$, in the case (a)

$$\liminf_{N\to\infty} 2^{-N}\beta^N p_N\left(y_{N+1} - 1\right) \geq 0$$

and our claim is proven. Now consider case (b). Taking limits in (33), we get

$$\sum_{t=1}^{\infty} 2^{-t}\left[v_t(1) - v\left(y_t\right)\right] - \sum_{t=0}^{\infty} 2^{-t}\beta^t p_t\left(y_t - y_{t+1}\right)$$

provided that the series on the right hand make sense. The first series diverges under our hypothesis. Thus, our claim will be true provided that the second series does not diverge too. On the other hand,

$$\begin{aligned} \sum_{t=0}^{\infty} 2^{-t}\beta^t p_t\left(y_t - y_{t+1}\right) &= \sum_{t=0}^{\infty} 2^{-t}\beta^t p_t\left(1 - y_{t+1}\right) - \sum_{t=1}^{\infty} 2^{-t}\beta^t p_t\left(1 - y_t\right) \\ &= \sum_{t=1}^{\infty} 2^{-t+1}\beta^{t-1}\left(p_{t-1} - 2^{-1}\beta p_t\right)\left(1 - y_t\right) \\ &= \sum_{t=1}^{\infty} 2^{-t}v_t'(1)\left(1 - y_t\right) \\ &\leq \sum_{t=1}^{\infty} 2^{-t}v_t'(1), \end{aligned}$$

where the third equality uses (32), and the desired result is proven.

Here we observe the superimposition of two kinds of bubbles: there are infinitely many equilibria and each of them might involve a bubble with respect to some state prices, as it can be shown by applying the argument of Example 2.

7 Concluding Remarks

When dealing with bubbles, it is unavoidable to run into some "paradoxes" related to the economics of infinity[9]. Although the economies studied in this paper prevent from well-known bubble-producing factors, like the presence of countably many finitely-lived agents or heterogeneous infinitely lived agents coupled with binding borrowing constraints, we have shown that there is a little room for additional kind of bubbles that economists should take into account, also in view of more general classes of intertemporal equilibrium models. Roughly speaking, we have met two paradoxes of infinity. The first one, the milder one, arises when an infinite number of states of the world is observable, at least at some trading date. The traditional first order condition (Euler equation), valid for a uniformly interior equilibrium, is no longer necessarily true. This has been discussed in Sections 5.2 and 5.3 and has led to the construction of the so-called bursting bubbles that violate the Euler equation. It must be underlined that this kind of bubbles is not related to the infinite-time horizon, they do survive in finite-horizon economies (Kamihigashi's example is just performed for a two-period economy). The second type requires a truly infinite economy and it has been illustrated through two examples in Subsection 6.3. Both are constructed by means of Petersburg assets, since they provide the simplest framework to handle truly infinite horizon economies where Kamihigashi sufficient condition of no-bubbles may fail. Example 5 is the most classical one because the states of the world are finite at every trading date. The paradox of infinity here is related to the already cited principle that an infinitely lived agent might gain by permanently reducing the asset holding, whenever a bubble occurred. This rule might be no longer true, for instance, if the agent has increasing relative risk-aversion through time[10].

The starting point of the present paper has been the formulation of the sufficient condition (14) that rules out bubbles. Theorem 2 is directly inspired by Kamihigashi [9], whose proof, in turn, draws on methods adopted by Ekeland and Scheinkman [5] in establishing necessary conditions of transversality at infinity for non-stochastic models of intertemporal optimization. A careful study of (14) has lead us to formulate a "generic" result of nonexistence of bubbles as well as to single out the rather important class of preferences exhibiting bounded relative risk-aversion for which bubbles never arise. Both results seem novel and should clarify to what extent bubbles may appear only under very special circumstances. Actually, a deeper analysis of (14) has opened the route leading to the classification, as well as to the construction of examples, of economies with positive bubbles.

One could ask whether the technique used in the proof of Theorem 2 can be successfully adopted in economies with heterogeneous infinitely lived agents as well. Since the most popular example of an economy with bubbles falls in this category, the question seems to be important enough. We have in mind Bewley's [1] consumption smoothing example with valued fiat money (see [11] and, more extensively, [16]). In such a setting, is not always possible to uniformly perturb downward the equilibrium trading plan, as it is not necessarily interior to the feasible region. In a companion paper (see [14]), we show that the original technique can be modified to obtain an interesting condition of transversality at infinity for the stochastic model, similar to that established by Kocherlakota [11] in the deterministic setting. Once again, the assumption of bounded relative risk-aversion plays a central role.

Section 6 accounts partially for the theory developed by Santos and Woodford [16] who intro-

[9] The intuition behind this can be easily understood in [19].

[10] Note that the focus on risk-aversion in our model has a respectable antecedent in Lucas' paper, where the relation between asset prices elasticity and relative risk aversion is pointed out.

duced an additional kind of bubbles arising from possible different valuations of the present value of future streams of wealth by means of state prices. Their Theorem 3.1 on non-existence of unambiguous bubbles must definitely be regarded as one of main contributions on bubbles fragility. Of course, our results here are hopeless far more limited, owing to the general setting we have chosen to work within. The roots of our treatment rest upon the asymptotic behavior of series (29) which is, in turn, closely related to the exclusion of Ponzi schemes. As far as we know, such an approach seems novel and the unique reference seems to be a paper by Sethi [18], whose analysis, however, is rather loosely related to ours. This series happens to exhibit a strong relationship with the exclusion of valuation bubbles. For instance, we have found a close link with our assumption of discounting A.3, which turns out to be a generalization of those formulated by Magill and Quinzii [13] and Santos and Woodford [16], within our setting. Furthermore, in Subsection 6.1 we briefly sketched an idea for a possible approach in ruling out bubbles by means of necessary conditions of optima for a deterministic economy. The extension to the stochastic setting has proven to be tough enough, as Theorem 6 requires strong assumptions for a relatively weak result. However, we believe that improvements in this direction are possible. For instance, it would be worth to investigating further whether some qualifications exist such that pointwise divergence of series (29) is sufficient to rule out bubbles.

8 Proofs

Proposition 1 requires two preliminary lemmas. Lemma 1 is established without proof as being rather familiar in constructing Euler equations, prescribing short-run conditions for optimality.

Lemma 1 *If* $\mathbf{p}$ *is an equilibrium price process, then necessarily*

$$u_{t-1}\left(c^*_{t-1}\right) - u_{t-1}\left(c_{t-1}\right) + \mathbf{E}_{t-1}\left[u_t\left(c^*_t\right) - u_t\left(c_t\right)\right] \geq 0$$

for all $t \geq 1$ *and for all* $c_{t-1}, y_t \geq 0$ $\mathcal{F}_{t-1}$*-measurable,* $c_t \geq 0$ $\mathcal{F}_t$*-measurable,* y_t *essentially bounded and provided that*

$$\begin{array}{c} c_{t-1} + p_{t-1} \cdot (y_t - \mathrm{e}) = c^*_{t-1} \\ c_t + p_t \cdot (\mathrm{e} - y_t) = d_t \cdot y_t + w_t \end{array} \tag{34}$$

Lemma 2 *Under A.1-2, if* $\mathbf{p}$ *is an equilibrium price process, then*

$$\left\{u'_{t-1}\left(c^*_{t-1}\right) p_{t-1} - \mathbf{E}_{t-1}\left[u'_t\left(c^*_t\right)\left(d_t + p_t\right)\right]\right\} \cdot y \geq 0 \tag{35}$$

for all $t \geq 1$ *and for all* $\mathcal{F}_{t-1}$*-measurable random vectors* $y(\omega) \in \mathbf{R}^k_+$ *such that:*

i) *y is essentially bounded,*

ii) $p_{t-1} \cdot y \leq \gamma c^*_{t-1}$ *, for some number* $\gamma > 0$*, depending on* y*.*

Proof. Fix $y(\omega)$, satisfying (i) and (ii), and consider the function:

$$J(\varepsilon) = u_{t-1}\left(c^*_{t-1}\right) - u_{t-1}\left(c_{t-1}\right) + \mathbf{E}_{t-1}\left[u_t\left(c^*_t\right) - u_t\left(c_t\right)\right]$$

for $0 \leq \varepsilon \leq \varepsilon_0$, with ε_0 small enough and where

$$\begin{aligned} y_t &= \mathrm{e} + \varepsilon y \\ c_{t-1} &= c_{t-1}^* - \varepsilon p_{t-1} \cdot y \\ c_t &= c_t^* + \varepsilon \left(p_t + d_t\right) \cdot y \end{aligned}$$

By construction, the triple (c_{t-1}, c_t, y_t) satisfies (34) and $y_t > 0$, $c_t > 0$. In force of (ii), $c_{t-1} > 0$ as long as $\varepsilon < \gamma^{-1}$. Clearly $J(0) = 0$ and, from Lemma 1, it must be $J(\varepsilon) \geq 0$. Moreover, $J(\varepsilon)$ is well defined because $J(\varepsilon) < +\infty$; this is true as $c_t \geq c_t^*$ and thus $u_t(c_t^*) - u_t(c_t) \leq 0$. Take a decreasing sequence $\varepsilon_n \to 0$ and consider the sequence $\varepsilon_n^{-1}[J(\varepsilon_n) - J(0)] \geq 0$. The limit $J_+^{'}(0)$ must be non-negative, provided it does exist. It is immediately seen that

$$J_+^{'}(0) = u_{t-1}^{'}\left(c_{t-1}^*\right) p_{t-1} \cdot y - \mathbf{E}_{t-1}\left[u_t^{'}\left(c_t^*\right)\left(p_t + d_t\right) \cdot y\right]$$

where the second addendum holds by the monotone convergence theorem, since the functions $\varepsilon_n^{-1}[u_t(c_t^*) - u_t(c_t)] \leq 0$ converge to $u_t^{'}(c_t^*)(p_t + d_t) \cdot y$ decreasingly in force of concavity of u_t. From $J_+^{'}(0) \geq 0$, (35) follows. ■

It is worth noticing how restriction (ii) be essential to derive Lemma 2 as well as Proposition 1.

Proof of Proposition 1. Fix an integer n and consider the event $A_n \in \mathcal{F}_{t-1}$ defined by $A_n = \left\{\omega; |p_{t-1}(\omega)| \leq n \text{ and } c_{t-1}^*(\omega) \geq n^{-1}\right\}$. As $n \to \infty$, $A_n \uparrow \Omega \setminus N$, where $\mu(N) = 0$. Consider the function $y(\omega) = \mathbf{1}_{A_n} v$, where $v \in \mathbf{R}_+^k$ is a fixed vector. This $\mathcal{F}_{t-1}$-measurable function meets assumptions (i) and (ii) of Lemma 2 and thus we can write:

$$\mathbf{1}_{A_n}\left\{u_{t-1}^{'}\left(c_{t-1}^*\right) p_{t-1} - \mathbf{E}_{t-1}\left[u_t^{'}\left(c_t^*\right)\left(p_t + d_t\right)\right]\right\} \cdot v \geq 0.$$

As vector $v \geq 0$ is arbitrary, it follows

$$u_{t-1}^{'}\left(c_{t-1}^*\right) p_{t-1} \geq \mathbf{E}_{t-1}\left[u_t^{'}\left(c_t^*\right)\left(p_t + d_t\right)\right]$$

for all $\omega \in A_n$. As n goes to infinity, it is true almost surely and this completes the proof. ■

Proof of Theorem 1. Observe that for any price sequence satisfying (1), $\mathbf{E}_0\left[u^{'}\left(c_t^*\right) p_t\right] < +\infty$, provided that $p_0 < +\infty$, as it has been assumed. This has two implications. First, $\mathbf{E}_0\left[u^{'}\left(c_t^*\right) p_t \cdot y\right] < +\infty$, for all essentially bounded functions $y(\omega)$. Second, it is easy to check to be $\mathbf{E}_s\left[u^{'}\left(c_t^*\right) p_t \cdot y\right] < +\infty$ for $s \geq 0$, as well. Let (c_t, y_t) be any feasible plan for a price process satisfying (1). Multiplying the budget constraint by $u_t^{'} \equiv u_t^{'}(c_t^*)$, we obtain

$$u_t^{'} c_t \leq u_t^{'} p_t \cdot (y_t - y_{t+1}) + u_t^{'} d_t \cdot y_t + u_t^{'} w_t$$

Taking the expected value and exploiting (1), we have

$$\begin{aligned} \mathbf{E}_{t-1}\left(u_t^{'} c_t\right) &\leq \mathbf{E}_{t-1}\left[\left(p_t + d_t\right) u_t^{'}\right] \cdot y_t - \mathbf{E}_{t-1}\left(u_t^{'} p_t \cdot y_{t+1}\right) + \mathbf{E}_{t-1}\left(u_t^{'} w_t\right) \\ &\leq u_{t-1}^{'} p_{t-1} \cdot y_t - \mathbf{E}_{t-1}\left(u_t^{'} p_t \cdot y_{t+1}\right) + \mathbf{E}_{t-1}\left(u_t^{'} w_t\right) \end{aligned}$$

Note that index t is taken greater than 0 and $\mathbf{E}_{t-1}\left(u_t^{'} p_t \cdot y_{t+1}\right)$ is finite, as y_{t+1} is assumed to be essentially bounded. Taking now the expected value $\mathbf{E}_0$ and summing up from $t = 1$ to $t = N$

$$\begin{aligned} \mathbf{E}_0 \textstyle\sum_{t=1}^N u_t^{'} c_t \leq \;& u_0^{'} p_0 \cdot y_1 - \mathbf{E}_0\left(u_N^{'} p_N \cdot y_{N+1}\right) \\ & + \mathbf{E}_0 \textstyle\sum_{t=1}^N u_t^{'} w_t \leq u_0^{'} p_0 \cdot y_1 + \mathbf{E}_0 \textstyle\sum_{t=1}^N u_t^{'} w_t. \end{aligned}$$

By adding the first term $u_0^{'}c_0 \le u_0^{'}p_0 \cdot (\mathrm{e} - y_1) + u_0^{'}d_0 \cdot \mathrm{e} + u_0^{'}w_0$, and by using (7), we get

$$\mathbf{E}_0 \sum_{t=0}^{N} u_t^{'} c_t \le u_0^{'} b_0 \cdot \mathrm{e} + \mathbf{E}_0 \sum_{t=0}^{N} u_t^{'} c_t^* + \mathbf{E}_0 \sum_{t=N+1}^{\infty} u_t^{'} d_t \cdot \mathrm{e}$$

that is true for all N, for any feasible consumption sequence and where b_0 is the price bubble at epoch 0. To conclude, from the concavity property $u_t(c_t^*) - u_t(c_t) \ge u_t^{'}(c_t^*)(c_t^* - c_t)$, it follows that

$$\mathbf{E}_0 \sum_{t=0}^{N} [u_t(c_t^*) - u_t(c_t)] \ge \mathbf{E}_0 \sum_{t=0}^{N} u_t^{'}(c_t^*)(c_t^* - c_t) \ge -u_0^{'} b_0 \cdot \mathrm{e} - \mathbf{E}_0 \sum_{t=N+1}^{\infty} u_t^{'} d_t \cdot \mathrm{e}$$

and, in force of (11),

$$\liminf_{N\to\infty} \mathbf{E}_0 \sum_{t=0}^{N} [u_t(c_t^*) - u_t(c_t)] \ge -(b_0 \cdot \mathrm{e})\, u_0^{'}$$

Therefore, if the prices agree with the fundamental values f_t, that is $b_0 = 0$, we obtain the desired property of optimality. ■

Proof of Theorem 2 It follows the same line of the proof of Lemma 4.1 in [9] and therefore we shall only sketch it. Consider the asset holding strategy $\mathbf{y}^1$ defined by $y_0^1 = \mathrm{e}$ and $y_t^1 = \mathrm{e} - \varepsilon v$ for $t \ge 1$, where v is a fixed vector in $\mathbf{R}_{++}^k$ and ε satisfies the condition $0 < \varepsilon v \le (1-\zeta)\mathrm{e}$, being ζ defined in (14). Let $\mathbf{c}^1$ be the corresponding consumption stream. Now, for $\alpha \in (0,1)$, define the plan $\mathbf{y}^\alpha = (1-\alpha)\,\mathrm{e} + \alpha \mathbf{y}^1$ with the relative consumptions $\mathbf{c}^\alpha = (1-\alpha)\mathbf{c}^* + \alpha \mathbf{c}$. By concavity,

$$\alpha^{-1}[u_t(c_t^*) - u_t(c_t^\alpha)] \le u_t(c_t^*) - u_t(c_t^1).$$

Since $c_t^\alpha \le c_t^*$, from $t \ge 1$ on, sums are increasing and, by taking the limit as $N \to \infty$ and then expectation, the following inequalities are true:

$$0 \le \mathbf{E}_0 \sum_{t=0}^{\infty} \alpha^{-1}[u_t(c_t^*) - u_t(c_t^\alpha)] \le \mathbf{E}_0 \sum_{t=0}^{\infty} [u_t(c_t^*) - u_t(c_t^1)] < +\infty$$

where the first is due to optimality of plan $\mathbf{c}^*$ (see (iii) of Definition 1), whilst the second is valid by (14). As $\alpha \downarrow 0$, the functions above increase, therefore, through repeated applications of the monotone convergence theorem, we get

$$u_0^{'}(c_0^*)\, p_0 \le \mathbf{E}_0 \sum_{t=1}^{\infty} u_t^{'}(c_t^*)\, d_t. \tag{36}$$

In view of (6), (36) yields $p_0 = f_0$, and the proof is complete because, from (9), $b_0 = 0$ implies $b_t = 0$ for all t. ■

Proof of Proposition 3. It is a simple variant of Theorem 2. It will be sufficient to consider the assets holding strategy $\mathbf{y}^i$ defined by $y_0^i = \mathrm{e}$ and $y_t^i = \mathrm{e} - \varepsilon \mathrm{e}_i$ for $t \ge 1$, where $\varepsilon > 0$ is sufficiently small and e_i is the vector with zero components but the i^{th}, which equals one. The corresponding consumption stream $\mathbf{c}^i$ is given by $c_0^i = c_0^* + \varepsilon p_0^i$ and $c_t^i = c_t^* - \varepsilon d_t^i$ for $t \ge 1$. The following steps closely follows those of the preceding proof. ■

Proof of Theorem 5. We claim the events

$$A_t = \left\{ d_t^i / p_t^i < \sigma_t \right\}$$

to be μ-negligible for all t. Arguing by contradiction, suppose that $\mu(A_s) > 0$ for some s. Then, picking ζ, relatively to A_s as established in A.3, one can rewrite this event as $A_s = \{\zeta d_s^i - \zeta \sigma_s p_s^i < 0\}$ or, equivalently, in vector notation

$$A_s = \left\{ \zeta d_s^i + p_s \cdot [(\mathrm{e} - \zeta \sigma_s \mathrm{e}_i) - \mathrm{e}] < 0 \right\} \tag{37}$$

where e_i denote the $\mathbf{R}^k$ vector with all null entries but the i^{th} equals 1.

We now construct a plan $\{\widetilde{c}_t, \widetilde{y}_t\}$ as follows: $\{\widetilde{c}_t, \widetilde{y}_t\} = \{c_t^*, \mathrm{e}\}$ for all $\omega \in \Omega$, if $t < s$ and for $\omega \notin A_s$ if $t \geq s$. If $\omega \in A_s$, then $\{\widetilde{c}_t, \widetilde{y}_t\} = \{c_s^* + \zeta d_s^i, \mathrm{e}\}$ and $\{\widetilde{c}_t, \widetilde{y}_t\} = \{c_t^* - \zeta \sigma_s d_t^i, \mathrm{e} - \zeta \sigma_s \mathrm{e}_i\}$ for $t \geq s + 1$. By using (37),

$$\begin{aligned} &(c_s^* + \zeta d_s^i) + p_s \cdot [(\mathrm{e} - \zeta \sigma_s \mathrm{e}_i) - \mathrm{e}] \leq c_s^* = d_s \cdot \mathrm{e} + w_s \qquad \text{and} \\ &c_t^* - \zeta \sigma_s d_t^i + p_t \cdot 0 = d_t \cdot (\mathrm{e} - \zeta \sigma_s \mathrm{e}_i) + w_t \qquad \text{for } t \geq s+1. \end{aligned}$$

and thus $\{\widetilde{c}_t, \widetilde{y}_t\}$ is feasible. By construction we have

$$\mathbf{E}_0 \sum_{t=0}^{N-1} [u_t(\widetilde{c}_t) - u_t(c_t^*)] = \mathbf{E}_0 \sum_{t=0}^{N-1} \mathbf{1}_{A_s} [u_t(\widetilde{c}_t) - u_t(c_t^*)]$$

for all $N \geq 1$. Taking the liminf, A.3 entails

$$\liminf_{N \to +\infty} \sum_{t=0}^{N-1} \mathbf{E}_0 [u_t(\widetilde{c}_t) - u_t(c_t^*)] > 0$$

which contradicts weak optimality of plan $\{c_t^*, \mathrm{e}\}$. Concluding, $\mu(A_t) = 0$, for all t and, consequently, $d_t^i / p_t^i \geq \sigma_t$ for almost all $\omega \in \Omega$. This implies our assert. ■

Proof of Theorem 6. Let us denote by γ^i the value of series $\sum_{t=1}^{\infty} d_t^i / p_t^i$ and by γ_n^i its finite truncation $\sum_{t=1}^{n} d_t^i / p_t^i$. First of all, observe that if the series in (22) diverges almost surely, then it diverges in probability. That means $\lim_{n \to \infty} \mu(\gamma_n^i \leq N) = 0$, for all N. Arguing by contradiction, if the series fails to be almost surely divergent then $\limsup_{n \to \infty} \mu(\gamma_n^i \leq M) > 0$, for some M. On the other hand, $\{\gamma_n^i \leq M\} \downarrow \{\gamma^i \leq M\}$, consequently $\mu(\gamma^i \leq M) > 0$. Since the events $A_n = \{\gamma_n^i \leq M\}$ are in $\mathcal{F}_n$, our assumption on $(\Omega, \mathbf{F}, \mu)$ entails that an s and some non-negligible element $B \in \mathcal{F}_s$ exist having the property $\gamma^i(\omega) \leq M$ on B. Notice also that in view of (27) and (23) he have $\widehat{y}_\infty^i \leq e^M \widehat{y}_0^i$. We claim that this leads to a contradiction. It suffices constructing the following plan

$$\widetilde{y}_t = \mathrm{e} - g_t \widehat{y}_t^i$$

where the functions $\widehat{y}_t^i$ of the self-financing plan $\{\widehat{y}_t^i\}$ are vectors having zero components but the i^{th}, the initial condition satisfies $\widehat{y}_0^i < e^{-M}$ and the sequence of functions $\{g_t\}$ is defined as: $g_t = 0$ for $0 \leq t \leq s$ and $g_t = \mathbf{1}_B$ for $t > s$. Since $B \in \mathcal{F}_s$ and $g_t \widehat{y}_t^i \leq \widehat{y}_t^i \leq \widehat{y}_\infty^i \leq e^M \widehat{y}_0^i < \mathrm{e}$, the plan $\widetilde{y}_t$ is feasible. The corresponding consumption sequence turns out to be

$$\widetilde{c}_t = c_t^* + (g_{t+1} - g_t)\, p_t \cdot \widehat{y}_{t+1}^i$$

Clearly, $\widetilde{c}_t = c_t^*$ for all $t \neq s$, while, $\widetilde{c}_s(\omega) = c_s^*(\omega)$ if $\omega \notin B$ and $\widetilde{c}_s(\omega) = c_s^*(\omega) + p_s^i(\omega) \cdot \widehat{y}_{s+1}^i(\omega)$ if $\omega \in B$. Note that $p_s^i(\omega) > 0$ on B. Hence, the plan $\{\widehat{y}_t^i\}$ finances a higher level of consumptions with positive probability and this contradicts optimality of $\mathbf{c}^*$. ■

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